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ISMMA Preprint Series

No. 1/2025

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A MULTIVARIATE CENTRAL LIMIT THEOREM FOR INDENTED QUANTUM RANDOM VARIABLES

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Abstract.We prove a multivariate CLT in T.Hasebe's indented probability theory, by generalizing our proofs for the CLT in M. Bożejko and R. Speicher's c-free probability theory, N. Muraki and Y.G.Lu's (anti-)monotone probability theory, and Hasebe's c-(anti-)monotone probability theory, and extending the combinatorial method exposed by F. Hiai and D. Petz [10], or A. Nica and R. Speicher [28], for the CLT in the setting of D.-V. Voiculescu's free probability theory. The joint moments of the corresponding quantum Gaussian family are described by an Isserlis-Wick type formula generalizing the ones derived in all these previous cases.

Key words: (ordered) non-crossing partition, (anti-)monotone partition, quantum probability space, non-commutative distribution, φ, ψ, θ-indentedness, ψ, θ-orderedness, Isserlis-Wick type formulae.

1. INTRODUCTION

D.-V. Voiculescu's seminal free probability theory (see, e.g., [33-35], but also [10,28] for further information) strongly motivated fundamental descoveries in the quantum probability (: QP) domain and its related fields. We send to, e.g., [5, 23, 29] (but also [11]), as an introduction into this domain. We remind, R. Speicher [31] and N. Muraki [26, 27] demonstrated there exist only five fundamental QP theories based on a quantum stochastic independence notion (involving a single state) emerging from an associative product of quantum probability spaces which possibly depends on the order of its factors. These five theories are: R. L. Hudson and K. R. Parthasarathy's Boson or Fermion probability theory, the free probability theory, Speicher and W. von Waldenfels' Boolean probability theory – corresponding to the tensor, free and Boolean product, which are not order-dependent; and, respectively, Muraki [24, 25] and Y.G.Lu's [21, 22] (anti-)monotone probability theory – corresponding to the (anti-)monotone product, which is order-dependent.

M. Bożejko and Speicher [3] unified the free and Boolean probability theory via their c-free independence concept (referring to a pair of states) arising from the c-free product [2, 3] of quantum probability spaces, which is associative, non-dependent on the order of its factors, and transfers its associativity to the free and Boolean products. Similarly, T. Hasebe [6, 7] unified the (anti-)monotone and Boolean probability theory by the c-(anti-)monotone independence notion (with respect to two states) emerging from his c-(anti-) monotone product; this being associative, dependent on the order of its factors, and transferring its associativity to the (anti-)monotone and Boolean products.

Moreover, by combining some c-free products, Hasebe [8] introduced an associative product for quantum probability spaces endowed with triples of states, in (*-) algebraic frame, initially named the indented product, which generalizes the (c-)free, (c-)(anti-) monotone and Boolean products, depends on the order of its factors and transfers its associativity to any of these seven aforementioned products. Consequently, the indented independence arising from this product is order-dependent: if a_1 and a_2 are indentedly independent random variables, it does not imply that a_2 and a_1 are, too.

Hasebe's indented probability theory (see [8] and the references therein) is an interesting and promising research topic generalizing and unifying the free, (anti-)monotone and Boolean probability theory and, moreover, the c-free and c-(anti-)monotone probability theory. Hasebe proved a univariate CLT in this frame for identically distributed random variables, with a Kesten (more generally, free Meixner) distribution (see, e.g., [11]) triple, as limit. In analogy, again, to Bożejko-Speicher theory, Muraki-Lu theory, and Hasebe's c-(anti-)monotone probability theory, the combinatorial structure of the indented independence is governed by the set of all (the ordered) non-crossing partitions, but it must distinguish not only between the outer and the inner blocks of such a partition, but also among its inner blocks according to their nearest covers.

In the present Note, we prove the multivariate CLT for φ,ψ,θ-indented (in particular, ψ,θ-ordered) random variables in Hasebe's theory, by generalizing, with respect to additional states, our elementary proofs from [15-17] of the CLT for φ,ψ-free, (anti-)monotone independent, and φ,ψ-(anti-)monotone independent random variables, in Bożejko-Speicher c-free probability theory, Muraki-Lu (anti-)monotone probability theory, and, respectively, Hasebe's [6, 7] c-(anti-)monotone probability theory, and extending the combinatorial moment method presented in [10] or [28] for the free CLT. Thus we derive an Isserlis-Wick type formula describing the joint moments of the corresponding multivariate quantum central limit distribution which generalizes all those formulae of this type obtained in these aforementioned cases. The setting is essentially that from [15, 17], but the simple random variables are this time more complicated, because the quantum probability space is endowed with a triple of states φ,ψ,θ and the peaks and bottoms are equally involved. Now, we focus on the occurrence of all these local extrema given by interval blocks in the ordered partition associated to a product of ψ,θ-centered φ,ψ,θ-indentedly independent random variables; via the weak independence in the sense of [4,12] once again. The alternative proof by cumulants is shorter. Other limit theorems can be proved. We will expose these elsewhere.

2. PRELIMINARIES

We repeat for the reader's convenience some well-known general information as in, e.g., [1, 8,11,15-17, 20,25-28], instead of sending directly to these references. (We abbreviate 'such that' by 's.t.', and 'with respect to' by 'w.r.t'). Let S be a finite totally ordered set (w.r.t. $<$). Denote by $P(S)$ the partitions of S; call blocks the non-empty subsets defining a partition. If S is a disjoint union of non-void subsets S_i, and $\pi\in P(S)$ s. t. $\pi=\cup\pi_i$, with some $\pi_i\in P(S_i)$, we write $\pi=\coprod\pi_i$. If, for instance, $S=\{s_1,...,s_n\}$, with $s_1<...<s_n$, we say $\pi\in P(S)$ is irreducible, when π does not factorize as $\pi_1\coprod\pi_2$, with $\pi_i\in P(S_i)$, where $S_1=\{s_1,...,s_p\}$ and $S_2=\{s_{p+1},...,s_n\}$ are disjoint sets.We call pairing a partition in which every block has exactly two elements. For $k,l\in S$, denote by $k\sim_\pi l$ the fact that k and l belong to the same block of $\pi\in P(S)$. Recall that a partition π is called crossing if there are $k_1<l_1<k_2<l_2$ in S s.t. $k_1\sim_\pi k_2\nsim_\pi l_1\sim_\pi l_2$; otherwise, π is non-crossing. When π is non-crossing, and V is a block of π, say V is inner, if there exists another block in π that covers V: i.e., there exist a block W of π, and $k,l\in W$, s. t. $k<v<l$, for all $v\in V$, denoting this by $W\prec V$; otherwise, say V is outer. For an inner block $V\in\pi$, the nearest cover in π of V is the block $W\in\pi$ covering V, for which there is no other block $W'\in\pi$ s.t. $W\prec W'\prec V$. Denote by $c_\pi(V)$ the nearest cover in π of V. Denote by $\mathrm{I}(\pi)$, and $\mathrm{O}(\pi)$ the inner, and, respectively, outer blocks of π. Recall that a non-crossing partition π is called an interval partition if $\mathrm{I}(\pi)$ is empty. Denote by $NC(S)$, $P_2(S)$, $NC_2(S)$ and $I_2(S)$ the non-crossing partitions, the pairings, the non-crossing pairings, and the interval pairings of S, respectively.

An ordered (coloured) partition of S is a partition $\pi=(P_1,...,P_r)$ of S endowed with an ordering (colouring) (: a permutation) of its blocks [20, 26]; s being the order (colour) of the block P_s. If $\pi\in P(S)$, there exist $|\pi|!$ ways to order (colour) π, where $|\pi|$ is the number of blocks of π. We symbol the block as $P_\bullet$ when its order (colour) is not specified. Denote by $OP(S)$ the ordered (coloured) partitions of S. Sometimes, we list together some partitions $\pi=(P_1,...,P_r)\in OP(S)$ and denote them as $\{P_{s_1},...,P_{s_r}\}\in OP(S)$, by putting their orders (colours) $1\le s_1,...,s_r\le r$ into a partition of the set $\{1,...,r\}$. Denote by $OP_{1,2}(S)$ the ordered (coloured) partitions of S for which every block has at most two elements, and by $ONC(S)$ the ordered (coloured) non-crossing partitions of S.

For a partition $\pi=(P_1,...,P_r)\in ONC(S)$, denote by $I_1(\pi)$ and $I_2(\pi)$ the set of the order-reflecting, respectively, order-reversing inner blocks in π (see, e.g. [8]); i.e.,

$I_1(\pi):=\{P_l\in\pi;$ the pair $P_k\prec P_l$ is order-reflecting$\,(:k<l)$ if $P_k=c_\pi(P_l)\}$, and

$I_2(\pi):=\{P_l\in\pi;$ the pair $P_k\prec P_l$ is order-reversing$\,(:k>l)$ if $P_k=c_\pi(P_l)\}=\mathrm{I}(\pi)\setminus I_1(\pi)$.

A monotone partition [20, 26] of S is a partition $\pi=(P_1,...,P_r)\in ONC(S)$ s.t. its ordering (colouring) is order-reflecting: for any pair of blocks $P_k\prec P_l$ in π, it holds $k<l$. If $\pi\in ONC(S)$ is not monotone, we say π is non-monotone. We denote by $M_2(S)$ the monotone pairings of S.

An anti-monotone partition [20, 26] of S is a partition $\pi=(P_1,...,P_r)\in ONC(S)$ s.t. its ordering (colouring) is order-reversing: for any pair of blocks $P_k\prec P_l$ in π, it holds $k>l$. If $\pi\in ONC(S)$ is not anti-monotone, we say π is non-anti-monotone. We denote by $AM_2(S)$ the anti-monotone pairings of S.

When S has m elements, abbreviate by $P_2(m)$, $NC_2(m)$, $I_2(m)$, $OP(m)$, $OP_2(m)$, $ONC_2(m)$, $M_2(m)$, and $AM_2(m)$ the pairings, non-crossing pairings, interval pairings, the ordered (coloured) partitions, pairings, non-crossing pairings, the monotone and anti-monotone pairings of S, respectively; and $OP_{1,2}(S)$ by $OP_{1,2}(m)$.The set $P_2(m)$ is empty if m is odd. Recall that each non-crossing partition of $\{1,...,m\}$ has at least an interval; i.e., a block of consecutive indices which may be a singleton (:block having a single element). Recall the cardinality of $P_2(2p)$ or $NC_2(2p)$ or $M_2(2p)$ (and also, $AM_2(2p)$) equals the corresponding moment of a standard Gauss, respectively, semi-circular Wigner or (by a factor of $p!$) arcsine distribution; i.e., $(2p)!!$, respectively the Catalan number $c_p:=(2p)!/p!(p+1)!$ or $(2p)!!$, too.

We consider a *- algebra as a (complex) associative algebra with an involution * (i.e. a conjugate linear anti-automorphism). A linear functional φ of a *- algebra A is positive if $\varphi(a^*a)\ge 0$, for all $a\in A$. If A is a (*-) algebra, and φ is a (positive) linear functional of A, we consider the unitization of A defined by $\tilde{A}:=\mathbb{C}\cdot 1\oplus A$, and the unitization of φ, denoted $\tilde{\varphi}$, given by $\tilde{\varphi}(\lambda\cdot 1\oplus a):=\varphi(a)+\lambda$, for any $\lambda\in\mathbb{C}$, $a\in A$. Let A be a (*-) algebra, and φ,ψ,θ be three states; i.e., linear (positive) functionals of A. We interpret (A,φ), (A,ψ) or (A,φ,ψ,θ) as quantum (*-) probability spaces, and the elements of A as quantum random variables in view of [33, 28]. When A has a unit 1 and the linear functionals involved φ are unital, i.e., $\varphi(1)=1$, we say the quantum probability space is unital. Otherwise, we say it is non-unital. Let I be an index set, $\mathbb{C}<\xi_i,i\in I>$ and $\mathbb{C}<\xi_i,i\in I>^\circ$ be the (*-) algebra with, respectively, without, a unit, freely generated by the complex field $\mathbb{C}$ and the non-commuting indeterminates $\xi_i, i\in I.$ Let (A,φ) be a quantum (*-) probability space, and $a=(a_i)_{i\in I}$ be such a random vector with all (self-adjoint) $a_i\in A$. The non-commutative joint distribution of a w.r.t. φ is $\varphi_a:=\varphi\circ\tau_a$, where $\tau_a:\mathbb{C}<\xi_i,i\in I>\to A$ is the unique unital (*-) homomorphism s.t. $\tau_a(\xi_i)=a_i$, if (A,φ) is unital, but $\tau_a : \mathbb{C} < \xi_i, i \in I >^{\circ} \to A$ is the unique (*-) homomorphism s.t. $\tau_a(\xi_i) = a_i$, if (A,φ) is not unital. The scalars $\varphi(a_{i_1}...a_{i_j})$ are viewed as the joint moments of a w.r.t. φ.

If $a_N = (a_N^i)_{i\in I}$ and $a = (a_i)_{i\in I}$ are random vectors in some quantum probability spaces (A_N, φ_N) and (A,φ), we say $(a_N)_N$ converges in distribution to a, denoting $a_N \xrightarrow{distr} a$, if for all $j \geq 1$, and all $i_1,...,i_j \in I$, $\lim_{N\to\infty} \varphi_N(a_N^{i_1}...a_N^{i_j}) = \varphi(a_{i_1}...a_{i_j})$. When $a \in A$ and $\varphi(a) = 0$, say a is centered w.r.t. φ, or φ-centered. When a is centered w.r.t. φ,ψ,θ, or w.r.t. ψ,θ, say it is φ,ψ,θ-centered, respectively ψ,θ-centered. For A unital and $a \in A$ (but, generally, $\varphi(a) \neq 0$), we center a w.r.t. φ, if we decompose $a = \varphi(a)\cdot 1 + a^{\circ}$ via the centering $a^{\circ} := a - \varphi(a)\cdot 1$ of a w.r.t. φ (see, e.g., [28, Notation 5.14]); 1 being here the unit of A. Thus, $a^{\circ} \in \ker\varphi$.

If I is totally ordered, $i_1,...,i_n \in I$ and $\{i_1,...,i_n\} = \{k_1,...,k_r\}$ with $k_1 < ... < k_r$, the ordered (coloured) partition corresponding to $j \mapsto i_j$ is $(P_1,...,P_r) \in OP(n)$ given by $P_j = \{s; i_s = k_j\}$ [20]. When $A_i \subset A$, $i \in I$ are subalgebras, and $w = a_1 \cdots a_n \in A$ is a random variable, s.t. all $a_j \in A_{i_j}$, for $i_1,...,i_n \in I$, the ordered (coloured) partition associated to w is that corresponding to $j \mapsto i_j$. We say w is crossing or non-crossing when this partition is crossing or non-crossing.

Let (A,φ,ψ,θ) be a unital quantum probability space, let I be a totally ordered set, let $A_i \subset A$, $i \in I$ be unital subalgebras, and $w = a_1 \cdots a_n \in A$ be a random variable, s.t. all $a_j \in A_{i_j}$ with $i_1,...,i_n \in I$. When there exists $2 \leq p < n$ with $i_{p-1} < i_p > i_{p+1}$ (respectively, $i_{p-1} > i_p < i_{p+1}$), we say $j \mapsto i_j$ has i_p as peak (respectively, bottom), and a_p is a peak (respectively, a bottom) in w. We say sometimes a peak is a local extremum of type 1, and a bottom is a local extremum of type 2. When $j \mapsto i_j$ has such local extrema, we say a_p is the left, respectively, right local extremum in w, if p is the minimum, respectively, the maximum of the set $\{2 \leq s \leq n-1; i_{s-1} < i_s > i_{s+1} \text{ or } i_{s-1} > i_s < i_{s+1}\}$.

We say the local extremum a_p is centered adequately to its type (for short, centered adequately) when a_p is ψ-centered, if it is a peak; respectively, θ-centered, if it is a bottom.

We center the local extremum a_p adequately to its type (for short, center adequately) i.e., $a_p = b_p \cdot 1 + a_p^{\circ}$, where $b_p \in \mathbb{C}$ and $a_p^{\circ} := a_p - b_p \cdot 1$, when we center a_p w.r.t. ψ, if it is a peak; respectively w.r.t. θ, if it is a bottom. In this case, we call a_p° the adequate centering of a_p, and name the random variables $a_1 \cdots a_{p-1}a_{p+1} \cdots a_n$ and $a_1 \cdots a_{p-1}a_p^{\circ}a_{p+1} \cdots a_n$ the subword obtained from w by excluding and, respectively, by centering a_p adequately to its type.

If $w = a_1 \cdots a_n \in A$, with $a_k \in A_{i_k}$ and $i_1,...,i_n \in I$, as above, has exactly k local extrema $e_k,...,e_1$, then we may expres $w = ue_kv_k \cdots e_1v_1$, with u, v_j as reduced words, being void or arbitrary products of $a_j \in A_{i_j}$; but u, v_j as non-void factors of w have no local extrema not centered adequately. By centering every e_j adequately to its type, i.e., $e_j = b_j \cdot 1 + e_j^{\circ}$, where $b_j \in \mathbb{C}$ is given by $b_j := \psi(e_j)$, if e_j is of type 1, but $b_j := \theta(e_j)$, if e_j is of type 2, and $e_j^{\circ} := e_j - b_j \cdot 1$, we expand w in the form below and name it *the pre-centered form* of w:

$$w = \sum_{j=1}^{k} b_j w^{(j)} + w^{\circ}, \text{ where } w^{\circ} := ue_k^{\circ}v_ke_{k-1}^{\circ}v_{k-1} \cdots e_1^{\circ}v_1;\ w^{(1)} := ue_kv_k \cdots e_2v_2v_1;$$

$w^{(j)} := ue_k v_k \cdots e_{j+1} v_{j+1} v_j e_{j-1}^{\circ} v_{j-1} \cdots e_1^{\circ} v_1$, for $2 \le j \le k-1$; and $w^{(k)} := uv_k e_{k-1}^{\circ} v_{k-1} \cdots e_1^{\circ} v_1$.

We say $w = a_1 \cdots a_n \in A$, with $a_k \in A_{i_k}$ and $i_1,...,i_n \in I$, is a simple random variable in (A,φ,ψ,θ) if w is reduced (i.e., $k \mapsto i_k$ has not intervals: $i_1 \ne i_2 \ne ... \ne i_n$), calling n the length of w, and w has no local extrema or w has only local extrema centered adequately to their types.

The next definition concerning the notion of φ,ψ,θ-indented independence (: shortly, φ,ψ,θ-indentedness) comes from [8], being inspired by [15-17]. If $\varphi=\psi$, say ψ,θ-ordered independence (shortly, ψ,θ- orderedness) instead of φ,ψ,θ-indented independence.

Definition 2.1 *Let* (A,φ,ψ,θ) *be a unital quantum probability space as above, and* $(1\in)A_i \subset A, i \in I$ *be unital subalgebras. The family* $(A_i)_{i\in I}$ *is* φ,ψ,θ*-indentedly independent (or* φ,ψ,θ*-indented, for short), if* $\varphi(a_1 \cdots a_n)=0$ *whenever* $n \ge 1$, $i_1 \ne ... \ne i_n$, *all* $a_k \in A_{i_k}$, $\varphi(a_n)=0$, *and* $\psi(a_p)=0$ *if* a_p *is a peak, but* $\theta(a_p)=0$ *if* a_p *is a bottom, for* $2 \le p < n$. *If* $A \supset S_i$, $i \in I$ *are subsets, then* $(S_i)_{i\in I}$ *is* φ,ψ,θ*-indentedly independent, if* $(A_i)_{i\in I}$ *is* φ,ψ,θ*-indentedly independent,* A_i *being the unital subalgebra of* A *generated by* S_i.□

In particular, Hasebe's ordered free independence/ordered-freeness w.r.t. (ψ,θ) according to [8], involves both the ψ,ψ,θ-indentedness and the θ,ψ,θ-indentedness; i.e., the ψ,θ-orderedness and the θ,ψ,θ- indentedness. Hasebe's indented independence w.r.t. (φ,ψ,θ), considered in [8], involves both the ordered free independence/ ordered-freeness w.r.t. (ψ,θ), and the φ,ψ,θ-indentedness.

Hasebe presented the conditions ii)-iii) below as being equivalent to the φ,ψ,θ-indentedness (see [8, Def.2.6 and Prop. 2.12].

Proposition 2.2 *Let* (A,φ,ψ,θ) *be a unital quantum probability space as above, and* $(1\in)A_i \subset A, i\in I$ *be unital subalgebras. The following are equivalent:*

i) $(A_i)_{i\in I}$ *is* φ,ψ,θ*-indentedly independent;*

ii) $\varphi(a_1 \cdots a_n)=0$ *whenever* $n \ge 1$, $i_1 \ne ... \ne i_n$, *all* $a_k \in A_{i_k}$, $\varphi(a_1)=0$, *and* $\psi(a_p)=0$ *if* a_p *is a peak, but* $\theta(a_p)=0$ *if* a_p *is a bottom, for* $2 \le p < n$;

iii) $\varphi(a_1 \cdots a_n)=0$ *whenever* $n \ge 1$, $i_1 \ne ... \ne i_n$, *all* $a_k \in A_{i_k}$, $\varphi(a_1)=0$, *and* $\psi(a_p)=0$ *if* $i_p > i_{p+1}$, *but* $\theta(a_p)=0$ *if* $i_p < i_{p+1}$, *for* $2 \le p < n$;

iv) $\varphi(a_1 \cdots a_n)=\varphi(a_1)\cdots\varphi(a_n)$ *whenever* $n \ge 1$, $i_1 \ne ... \ne i_n$, *all* $a_k \in A_{i_k}$, *and* $\psi(a_p)=0$ *if* $i_p > i_{p+1}$, *but* $\theta(a_p)=0$ *if* $i_p < i_{p+1}$, *for* $2 \le p < n$. □

We could demonstrate the statements in this Note by iii) or iv), too; but we do not use these in the sequel.

We add iv) above to clarify how the indented product of quantum probability spaces realizing the φ,ψ,θ-indentedness is a special case of the product presented in [4, 12]. (see [8, Remark 2.9].)

The notion of φ,ψ,θ-indentedness extends the notions of (φ,ψ-)freeness, (φ,ψ-)(anti-)monotone independence and Boolean independence.

Remarks 2.3 1)*The* φ,ψ,ψ*-indentedness is the* φ,ψ*-freeness* [15]*; the* φ,φ,φ*-indentedness (or the* φ,φ*-orderedness) is Voiculescu's freeness w.r.t.* φ [33-35, 1, 5, 10-17, 20, 28]. *The conditionally free* *(or c-free, for brevity) independence w.r.t.* (φ, ψ)*, considered in* [2, 6-8,19], *involves both the* φ, ψ*-freeness and the freeness w.r.t* ψ.

2) *The* φ, ψ, θ*-indentedness generalizes the* φ, ψ*-(anti-)monotone independence* [17] *and the Boolean independence* [32], *too. The* ψ, ψ*-(anti-)monotone independence is Muraki-Lu's (anti-)monotone independence w.r.t.*ψ [6-9, 11, 16, 17, 20-22, 24-27]. *The conditionally monotone (or c-monotone, for brevity) independence w.r.t.* (φ, ψ)*, considered in* [6-8,19], *involves both the monotone independence w.r.t.* ψ*, and the* φ, ψ *- monotone independence.*

More exactly, the following hold true.

Let (A,φ,ψ) *be an arbitrary quantum probability space, and* $A_i \subset A$, $i \in I$ *be a family of subalgebras of* A. *Let* $(\tilde{A},\tilde{\varphi},\tilde{\psi},\delta)$ *and* $1 \in \tilde{A}_i \subset \tilde{A}, i \in I$ *be the unital quantum probability space and, respectively, the family of unital subalgebras in this, consisting of the corresponding unitizations*; δ *being the unitization of the functional* 0 *of* A.

If $(\tilde{A}_i)_{i\in I}$ *is* $\tilde{\varphi}, \tilde{\psi}, \delta$*-indentedly independent in* $(\tilde{A},\tilde{\varphi},\tilde{\psi},\delta)$*, then* $(A_i)_{i\in I}$ *is* φ, ψ*-monotone independent in* (A,φ,ψ).

If $(\tilde{A}_i)_{i\in I}$ *is* $\tilde{\varphi}, \delta, \tilde{\psi}$*-indentedly independent in* $(\tilde{A},\tilde{\varphi},\tilde{\psi},\delta)$*, then* $(A_i)_{i\in I}$ *is* φ, ψ*-anti-monotone independent in* (A,φ,ψ).

If $(\tilde{A}_i)_{i\in I}$ *is* $\tilde{\varphi}, \delta, \delta$*-indentedly independent in* $(\tilde{A},\tilde{\varphi},\delta,\delta)$*, then* $(A_i)_{i\in I}$ *is Boolean independent in* (A,φ).□

We prefer to maintain Hasebe's initial denominations concerning his notions of three (two)-state quantum independence and product of quantum probability spaces realizing it.

3. JOINT MOMENTS OF φ, ψ, θ-INDENTED QUANTUM RANDOM VARIABLES

In this section I will be a totally ordered set, (A,φ,ψ,θ) will be a unital quantum probability space as before, and $A_i \subset A$, $i \in I$ will be a family of φ, ψ, θ-indentedly independent unital subalgebras of A.

The following lemma will be used several times in the sequel.

Lemma 3.1 *Let* $w = a_1 \cdots a_n \in A$ *be reduced, s.t. every* $a_j \in A_{i_j}$.

1) *If* w *(: the map* $j \mapsto i_j$*) has no local extrema, then* $\varphi(w) = \varphi(a_1)\cdots\varphi(a_n)$.

2) *If* w *is a simple random variable in* (A,φ,ψ,θ)*, and* $\varphi(a_1) = 0$ (respectively $\varphi(a_n) = 0$)*, then* $\varphi(w) = 0$.□

We observe $(A_i)_{i\in I}$ is weakly independent in (A,φ) in the sense of [4,12]; remind the weak-independence has the meaning below.

Definition 3.2 *Let* (B,ω) *be a unital quantum probability space as above and* $B_i \subset B, i \in I$ *be unital subalgebras. The family* $(B_i)_{i\in I}$ *is weakly independent in* (B,ω)*, if* $\omega(x_1 ... x_n) = \omega(x_1 ... x_p)\omega(x_{p+1} ... x_n)$*, for all* $n > p \geq 1$*, all* $i_j \in I$*, all* $x_j \in B_{i_j}$, s.t. *the sets* $\{i_1,...,i_p\}$ *and* $\{i_{p+1},...,i_n\}$ *are disjoint. If* $B \supset S_i$, $i \in I$ *are subsets, then* $(S_i)_{i\in I}$ *is weakly independent, if* $(B_i)_{i\in I}$ *is weakly independent*; B_i *being the unital subalgebra of* B *generated by* S_i.□

The second lemma extends [16, Lemma 3.2] and [17, Lemma 3.3].

Lemma 3.3 $(A_i)_{i\in I}$ *is weakly independent in* (A,φ); *i.e.,*

$\varphi(a_1\cdots a_n)=\varphi(a_1\cdots a_p)\varphi(a_{p+1}\cdots a_n)$, *for all* $n>p\geq 1$, *all* $i_j\in I$, *all* $a_j\in A_{i_j}$, s.t. *the sets* $\{i_1,...,i_p\}$ *and* $\{i_{p+1},...,i_n\}$ *are disjoint.*

Proof. It suffices to suppose $i_1\neq i_2\neq ...\neq i_n$. If $j\mapsto i_j$ has no local extrema, the statement follows from Lemma 3.1. Otherwise, for $n=3$, it results by centering the single local extremum a_2 adequately to its type, via Lemma 3.1 and the assertion for $n=2$.

Let $n>3$. Suppose the statement true for any $a_1\cdots a_s\in A$ having local extrema and the length $s<n$. Verify it for $w=a_n\cdots a_1\in A$ as below to conclude by induction.

Consider the case when a_{p+1} and a_p are not local extrema; and thus $w=a_nxa_{p+1}a_pya_1$ with $x=ue_rv_r\cdots e_qv_q$, and $y=ve_{q-1}v_{q-1}\cdots e_1v_1$ where $r>q>1$; u,v,v_l are void or arbitrary products of $a_j\in A_{i_j}$; and $e_1,...,e_q,...,e_r$ are all the local extrema in w. But u,v,v_l as non-void factors in x or y have no local extrema.

Center a_1 w.r.t. φ and each local extremum e_j adequately to its type: $a_1=b\cdot 1+a_1^\circ$, and $e_j=b_j\cdot 1+e_j^\circ$, with scalars $b:=\varphi(a_1)$, and $b_j\in\mathbb{C}$. Then expand $y=\sum_{l=1}^{q-1}b_ly^{(l)}+y^\circ$ in its pre-centered form (with $y^{(l)}$ and y° as in Preliminaries), to get

$$w=ba_nxa_{p+1}a_py+\sum_{l=1}^{q-1}b_la_nxa_{p+1}a_py^{(l)}a_1^\circ+a_nxa_{p+1}a_py^\circ a_1^\circ .$$

By expanding also $x=\sum_{k=q}^{r}b_kx^{(k)}+x^\circ$ in its pre-centered form (with corresponding $x^{(k)}$ and x° defined as in Preliminaries), remark the term $a_nxa_{p+1}a_py^\circ a_1^\circ=\sum_{k=q}^{r}b_ka_nx^{(k)}a_{p+1}a_py^\circ a_1^\circ+a_nx^\circ a_{p+1}a_py^\circ a_1^\circ$ belongs to the kernel of φ; due to the inductive hypothesis and Lemma 3.1, for any term $a_nx^{(k)}a_{p+1}a_py^\circ a_1^\circ$, having the length inferior to n, and to Lemma 3.1, too, for the term of maximum length $a_nx^\circ a_{p+1}a_py^\circ a_1^\circ$ which is a simple random variable in (A,φ,ψ,θ).

Since every term $a_nxa_{p+1}a_py^{(l)}a_1^\circ$ has the length inferior to n, the inductive hypothesis implies then

$$\varphi(w)=\varphi(a_nxa_{p+1})[b\varphi(a_py)+\sum_{l=1}^{q-1}\varphi(a_pb_ly^{(l)}a_1^\circ)]=\varphi(a_nxa_{p+1})[\varphi(ba_py)+\varphi(a_p(y-y^\circ)a_1^\circ)]=$$

$=\varphi(a_nxa_{p+1})\varphi(a_pya_1)$, hence the statement for n; finally, because $a_py^\circ a_1^\circ$ is a simple random variable in (A,φ,ψ,θ) being in the kernel of φ, by Lemma 3.1, again.

The other cases (: when a_{p+1} and a_p are both local extrema, or only one of them is a local extremum, in w), are variations on the theme above. We let this exercise to the reader.□

For $w=a_1\cdots a_n\in A$ s.t. every $a_j\in A_{i_j}$, we say w has a_k as singleton when $i_j\neq i_k$, for any $j\neq k$.

The following statement generalizes [15, Lemma 3.2], [16, Lemma 3.3], and [17, Lemma 3.4].

Lemma 3.4 *Let* $w = a_1 \cdots a_n \in A$*, s.t. every* $a_j \in A_{i_j}$*, and* w *has a singleton* a_k *which is* φ, ψ, θ*-centered. Then* $\varphi(w) = 0$.

Proof. It suffices to suppose w is reduced. If $k \in \{1, n\}$, the assertion follows by the weak-independence (i.e., Lemma 3.3), and the φ-centeredness of a_k. It remains to consider $2 \le k \le n-1$. If $j \mapsto i_j$ has no local extrema, Lemma 3.1 and the φ-centeredness of a_k, again, imply $\varphi(w) = \varphi(a_1)\cdots\varphi(a_k)\cdots\varphi(a_n) = 0$, for all $n \ge 3$.

Otherwise, for $n = 3$, the single local extremum a_k being centered adequately to its type, the assertion results by Lemma 3.1, via the centering of a_n w.r.t. φ and the φ-centeredness of a_k, again.

Let $n > 3$. Suppose the statement valid for any $a_1 \cdots a_r \in A$ having local extrema and the length $r < n$; check it for $w = a_1 \cdots a_n \in A$, as below, to conclude by induction.

Consider w has exactly p local extrema $e_1, ..., e_p$ (including, possibly, a_k).

Thus, we may express $w = a_1 x a_n$ with $x = u e_p v_p \cdots e_1 v_1$, where u, v_l are void or arbitrary products of $a_j \in A_{i_j}$. But u, v_l as non-void factors in x have no local extrema.

Center a_n w.r.t. φ, i.e., $a_n = b_n \cdot 1 + a_n^\circ$, where $b_n := \varphi(a_n)$, and each local extremum e_j adequately to its type (: $e_j = b_j \cdot 1 + e_j^\circ$, with scalars $b_j \in \mathbb{C}$), and develop then x in the pre-centered form $x = \sum_{j=1}^{p} b_j x^{(j)} + x^\circ$, to get $w = b_n a_1 \cdots a_k \cdots a_{n-1} + \sum_{j=1}^{p} b_j a_1 x^{(j)} a_n^\circ + a_1 x^\circ a_n^\circ$ (with $x^{(j)}$ and x° defined as in Preliminaries); where $a_1 x$ and all the terms $a_1 x^{(j)} a_n^\circ$ have the length inferior to n.

It remains only to note (via, possibly, the ψ, θ-centeredness of a_k, if needed) the kernel of φ contains $a_1 x$, each term $a_1 x^{(j)} a_n^\circ$ (due to the induction hypothesis); and the term of maximum length $a_1 x^\circ a_n^\circ$, too, by Lemma 3.1, as a simple random variable in $(A, \varphi, \psi, \theta)$.□

The combinatorial structure of the φ, ψ, θ-indentedness is more complicated than that of the φ, ψ-freeness [15] or the φ, ψ-(anti-)monotone independence [17]; and the partitions involved in the lemmata below are now many more. However, with our approach, their tratement is easy.

We illustrate the *next statement* by the following classes of partitions π_j in $OP_{1,2}(m)$ associated to $a_1 x c_r y a_m = w$.

Examples 3.5 1) If $m = 5$, let $\pi_1 = ((4)_1, (1,5)_2, (2,3)_3)$ and $\pi_2 = ((2,3)_1, (1,5)_2, (4)_3)$ being ordered non-crossing, but non-(anti-)monotone. For each of them, the unique non-centered local extremum $e := a_2 a_3 \in A_{i_2}$ arises, in the reduced form of $w = a_1 x c a_m$, from the interval block $(2,3)$. as a peak, for π_1, and a bottom, for π_2; denoting $x := e$, and $c := a_4$. The subwords obtained from w, by excluding or by centering e adequately to its type, are both simple random variable in $(A, \varphi, \psi, \theta)$; so, $\varphi(w) = 0$, for both π_j, by Lemma 3.1.

2) For $m = 7$, let π be any of the following ordered crossing partitions :

i) $\{(2,3)_s, (6)_t, (1,5)_u, (4,m)_v\}, s,t \in \{1,2\}, u,v \in \{3,4\}$; $\{(2,3)_s, (6)_t, (1,5)_u, (4,m)_v\}$ $s,t \in \{1,4\}$, $u,v \in \{2,3\}$; $\{(1,5)_u, (4,m)_v (2,3)_s, (6)_t\}, u,v \in \{1,2\}, s,t \in \{3,4\}$;

ii) $\{(1,6)_1, (4)_2, (2,3)_s, (5,m)_t\}, s,t \in \{3,4\}$; $\{(2,3)_s, (5,m)_t, (1,6)_u, (4)_v\}, s,t \in \{1,2\}, u,v \in \{3,4\}$; $((4)_1, (5,m)_2, (1,6)_3, (2,3)_4)$, $((2,3)_1, (1,6)_2, (5,m)_3, (4)_4)$;

For each of them, the unique non-centered local extremum $e := a_2 a_3 \in A_{i_2}$ arises, in the reduced form of $w = a_1 x c y a_m$, from the interval block $(2,3)_\cdot$. The subwords obtained from w, by excluding or by centering e adequately to its type, are both simple random variable in (A,φ,ψ,θ); so, $\varphi(w) = 0$, by Lemma 3.1.

3) For $m = 7$, let π be any of the ordered partitions listed below:

iii) $\{(3,4)_s,(6)_t,(1,5)_u,(2,m)_v\}$, $s,t \in \{1,4\}$, $u,v \in \{2,3\}$; $\{(3,4)_s,(6)_t,(1,5)_u,(2,m)_v\}$ $s,t \in \{1,2\}$, $u,v \in \{3,4\}$; $\{(1,5)_u,(2,m)_v,(3,4)_s,(6)_t\}$, $u,v \in \{1,2\}$, $s,t \in \{3,4\}$;

iv) $\{(3,4)_s,(1,6)_t,(2,m)_u,(5)_v\}$ $s,t \in \{1,2\}$, $u,v \in \{3,4\}$; $\{(2,m)_u,(5)_v,(3,4)_s,(1,6)_t\}$ $u,v \in \{1,2\}$, $s,t \in \{3,4\}$; $((3,4)_1,(2,m)_2,(1,6)_3,(5)_4)$, $((5)_1,(1,6)_2,(2,m)_3,(3,4)_4)$;

v) $\{(5)_1,(2,6)_2,(3,4)_s,(1,m)_t\}$ $s,t \in \{3,4\}$; $\{(3,4)_s,(1,m)_t,(2,6)_3,(5)_4\}$ $s,t \in \{1,2\}$; $\{(3,4)_1,(2,6)_2,(1,m)_s,(5)_t\}$ $s,t \in \{3,4\}$; $\{(1,m)_s,(5)_t,(2,6)_3,(3,4)_4\}$ $s,t \in \{1,2\}$ [non-crossing];

These are crossing, except for v) which are non-(anti-)monotone non-crossing partitions.The unique non-centered local extremum e arises, in the reduced form of $w = a_1 x c y a_m$, from the interval block $(3,4)_\cdot$. The subword obtained from w, by centering e, adequately to its type, is a simple random variable in (A,φ,ψ,θ), in each case. The subword obtained from w, by excluding e, adequately to its type, is a simple random variable in (A,φ,ψ,θ) in every crossing case; and, in the non-crossing cases its computation reduces to the above examples for $m = 5$. So, $\varphi(w) = 0$, by Lemma 3.1, always.

4) For $m = 7$, let π be any of the non-(anti-)monotone non-crossing partitions $\{(2,3)_s,(1,m)_u,(4,5)_v,(6)_t\}$, $s,t \in \{1,4\}$, $u,v \in \{2,3\}$.

For each of them, the unique non-centered local extremum $e := a_2 a_3 \in A_{i_2}$, in the reduced form of $w = a_1 x c a_m$, arises from the interval block $(2,3)_\cdot$; with $c := a_6$, $x := e a_4 a_5$. But the interval block $(4,5)_\cdot$ does not give a local extremum in w. The reduced subword obtained from w, by centering e, adequately to its type, is a simple random variable in (A,φ,ψ,θ), in each case. Moreover, the subword obtained from w, by excluding e, adequately to its type, is a simple random variable, too; except for $u = 3$, $v = 2$, $t = 4$ and $u = 2$, $v = 3$, $t = 1$, when its computation reduces to the examples before for $m = 5$, and $a_4 a_5 \in A_{i_4}$ becomes a local extremum in the reduced form of this subword. Hence, $\varphi(a_1 x c a_m) = 0$, always, via Lemma 3.1.

5). For $m = 7$, let π be any of the following ordered non-crossing partitions

ix) $\{(6)_1,(2,5)_2,(3,4)_s,(1,m)_t\}$, $s,t \in \{3,4\}$; $\{(3,4)_s,(1,m)_t,(2,5)_3,(6)_4\}$, $s,t \in \{1,2\}$;

x) $\{(3,4)_s,(6)_t,(2,5)_3,(1,m)_4\}$ $s,t \in \{1,2\}$ [anti-monotone];

xi) $\{(6)_1,(3,4)_s,(1,m)_t,(2,5)_4\}$, $s,t \in \{2,3\}$; $\{(3,4)_1,(2,5)_s,(1,m)_3,(6)_t\}$ $s,t \in \{2,4\}$; $((2,5)_1,(3,4)_2,(1,m)_3,(6)_4)$.

These are non-(anti-)monotone, except for x) which are anti-monotone. For each of them, the unique non-centered local extremum $e := a_3 a_4 \in A_{i_3}$ arises in the reduced form of $w = a_1 x c a_m$, from the interval block $(3,4)_\cdot$, with $x := a_2 e a_5$; and $c := a_6$ as peak or bottom, adequately centered. The reduced subword obtained from w, by centering e, adequately to its type, is a simple random variable in (A,φ,ψ,θ), in all cases. For ix)-x), the reduced subword obtained from w, by excluding e, adequately to its type, is a simple random variable in (A,φ,ψ,θ), too. In rest, the value of φ corresponding to this subword reduces to the examples before for $m = 5$, again. Hence, $\varphi(a_1 x c a_m) = 0$, always, via Lemma 3.1. □

The next statement generalizes [15, Lemmata 3.5-3.6].

Lemma 3.6 *Let* $w = a_1 x c y a_m \in A$, *s.t.:* $a_1 \in A_{i_1}$, $a_m \in A_{i_m}$; x *and* y *are possibly void products of* $a_j \in A_{i_j}$ *with* $\psi(a_j) = \theta(a_j) = 0$; $c \in A_{i_r}$ *is an adequately centered local extremum singleton in w;* $c y a_m$ *(respectively,* $a_1 x c$*), under its reduced form, is a simple random variable in*

(A,φ,ψ,θ) *;* $\varphi(a_m)=0$ *(respectively,* $\varphi(a_1)=0$*); and the ordered partition associated to* a_1xya_m *is a pairing.*

Then $\varphi(w)=0$*, whenever* a_1xya_m *is crossing, or* $i_1=i_m$ *and* xy *is non-crossing or void.*

Proof. In light of Lemma 3.3 (the weak independence) and Lemma 3.1, it suffices to consider the ordered partition π associated to w is irreducible, and x has (under its reduced form) local extrema arising from some interval blocks of π. The second part of this lemma, when a_1xc is simple and $\varphi(a_1)=0$, results by an analogous argument.

For $m=5$, see Examples 3.5. For $m=7$, it is easy to note there are involved only the ordered partitions $\pi\in OP_{1,2}(m)$ 1)-4) listed below, besides the partitions in Examples 3.5 before.

1) The crossing partitions
$\{(1,6)_u,(4,m)_v,(2,3)_s,(5)_t\}$, $s,t\in\{1,2\}$, $u,v\in\{3,4\}$; $\{(2,3)_s,(5)_t,(1,6)_u,(4,m)_v\}$, $s,t\in\{1,4\}$, $u,v\in\{2,3\}$; $\{(2,3)_s,(5)_t,(1,6)_u,(4,m)_v\}$, $s,t\in\{1,2\}$, $u,v\in\{3,4\}$, and
the non-(anti-)monotone non-crossing partitions
$\{(2,3)_s,(5)_t,(1,m)_u,(4,6)_v\}$, $s,t\in\{1,2\}$, $u,v\in\{3,4\}$; $\{(2,3)_s,(5)_t,(1,m)_u,(4,6)_v\}$, $s,t\in\{3,4\}$, $u,v\in\{1,2\}$?; $\{(1,m)_s,(5)_t,(4,6)_3,(2,3)_4\}$, $s,t\in\{1,2\}$; $\{(2,3)_1,(4,6)_2,(1,m)_s,(5)_t\}$, $s,t\in\{3,4\}$; $((4,6)_1,(5)_2,(1,m)_3,(2,3)_4)$, $((5)_1,(4,6)_2,(1,m)_3,(2,3)_4)$, $((2,3)_1,(1,m)_2,(5)_3,(4,6)_4)$, $((2,3)_1,(1,m)_2,(4,6)_3,(5)_4)$.

For each of them, the unique non-centered local extremum $e:=a_2a_3\in A_{i_2}$ arises, in the reduced form of $w=a_1xcya_m$, from the interval block $(2,3)_{\cdot}$; and $x:=ea_4$, $c:=a_5$. The subwords obtained from w by excluding or by centering e, adequately to its type, are both simple random variable in (A,φ,ψ,θ); so, $\varphi(w)=0$, by Lemma 3.1.

2) The crossing partitions
$\{(1,3)_u,(6)_v,(4,5)_s,(2,m)_t\}$ $u,v\in\{1,2\}$, $s,t\in\{3,4\}$; $\{(4,5)_s,(2,m)_t,(1,3)_u,(6)_v\}$ $s,t\in\{1,2\}$, $u,v\in\{3,4\}$; $((4,5)_1,(1,3)_2,(2,m)_3,(6)_4)$, $((6)_1,(2,m)_2,(1,3)_3,(4,5)_4)$
and the non-(anti-)monotone non-crossing partitions
$((4,5)_1,(2,3)_2,(1,m)_3,(6)_4)$ and $((6)_1,(1,m)_2,(2,3)_3,(4,5)_4)$.

For these partitions, the unique non-centered local extremum $e:=a_4a_5\in A_{i_4}$ arises, in the reduced form of $w=a_1xca_m$, from the interval block $(4,5)_{\cdot}$; with $x:=a_2a_3e$, $c:=a_6$ in both non-crossing cases (when $a_2a_3\in A_{i_2}$ is not a local extremum in w). The reduced subword obtained from w by centering e, adequately to its type, is a simple random variable in (A,φ,ψ,θ), in all cases; so is also the subword obtained from w by excluding e, adequately to its type, in the crossing cases. But, in the non-crossing cases, a_2a_3 becomes a local extremum in the reduced form of the subword obtained from w by excluding e, adequately to its type; and note, the value of φ corresponding to this subword reduces to the above Examples 3.5 for $m=5$. Thus, $\varphi(a_1xca_m)=0$ in all cases, via Lemma 3.1.

3) The ordered non-crossing partitions
$\{(1,m)_u,(2,5)_v,(3,4)_s,(6)_t\}$, $u,v\in\{1,2\}$, $s,t\in\{3,4\}$ and $((6)_1,(1,m)_2,(2,5)_3,(3,4)_4)$.

For each of them, the unique non-centered local extremum $e:=a_3a_4\in A_{i_3}$ arises as a peak, in the reduced form of $w=a_1xca_m$, from the interval block $(3,4)_{\cdot}$, with $x:=a_2ea_5$; and $c:=a_6$ as peak or bottom adequately centered. The reduced subword obtained from w by centering e, adequately to its type, is a simple random variable in (A,φ,ψ,θ), in all cases. For the two monotone cases (: $u=1$, $v=2$), the reduced subword obtained from w by excluding e, adequately to its type, is a simple random variable in (A,φ,ψ,θ), too. As for the rest, for the non-(anti-)monotone partitions, the value of φ corresponding to this subword reduces to the Examples 3.5 before for $m=5$, again. Thus, $\varphi(a_1xca_m)=0$ in all cases, via Lemma 3.1.

4) Finally, the ordered non-crossing partitions:
$\{(1,m)_u,(4,5)_v,(2,3)_s,(6)_t\}, u,v\in\{1,2\}, s,t\in\{3,4\}$ (monotone) and $\{(2,3)_s,(6)_t,(1,m)_u,(4,5)_v\}$,
$s,t\in\{1,2\}, u,v\in\{3,4\}$ (anti-monotone).

For them, the interval blocks $(2,3)$. and $(4,5)$. give the non-centered local extrema e_2, e_1 of type 1, and 2, in the monotone case; but of type 2, and 1, in the anti-monotone case. By centering e_2, e_1 adequately to their type $e_j=b_j\cdot 1+e_j^\circ$, where the scalars $b_j\in\mathbb{C}$ are given by $b_2:=\psi(e_2)$, and $b_1:=\theta(e_1)$, for the monotone case , but $b_2:=\theta(e_2)$, and $b_1:=\psi(e_1)$, for the anti-monotone case, and $e_j^\circ:=e_j-b_j\cdot 1$, we expand $x:=e_2e_1 = x_\circ+x_1+x_2+x^\circ$, with $x_\circ:=b_1b_2$, $x_1:=b_1e_2^\circ$, $x_2:=b_2e_1^\circ$, and $x^\circ:=e_2^\circ e_1^\circ$, and note all random variables $a_1x_\circ ca_n$, $a_1x_1ca_n$, $a_1x_2ca_n$ and $a_1x^\circ ca_n$ are simple in (A,φ,ψ,θ); with $c:=a_6$ as peak or bottom, in the monotone, and, respectively, anti-monotone case. Thus, they are in $\ker\varphi$ by Lemma 3.1, once again. Hence, $\varphi(a_1xca_m)=0$, always.

Letting $m>7$, suppose the assertion true for any word a_1xcya_p s.t. the ordered subpartition of π associated to a_1xya_p belongs to $OP_2(p-1)$ with $p<m$, and verify it for m as follows. Consider $w=a_1xcya_m$ and the ordered subpartition of π associated to a_1xya_m belonging to $OP_2(m-1)$. We may proceed in a similar way with [15, Lemma 3.6]. Assume the ordered subpartition of π associated to x has exactly k interval blocks giving local extremum singletons $e_k,...,e_1$, so that $x=ue_kv_k\cdots e_1v_1$, with u,v_j as reduced words being void or arbitrary products of $a_j\in A_{i_j}$; else, the reasoning is similar. Center e_j adequately to its type $e_j=b_j\cdot 1+e_j^\circ$, (with $b_j\in\mathbb{C}$, etc.) to expand in the pre-centered form $x=\sum_{j=1}^{k}b_jx^{(j)}+x^\circ$ (with $x^{(j)}$ and x° defined as in Preliminaries).

The ordered subpartition of π associated to $a_1x^{(1)}ya_m$ belongs to $OP_2(m-3)$, and $x^{(2)},...,x^{(k)}$ may be expressed as algebraic sums (with ± 1 as coefficients) of random variables $\bar{x}$ having the same generic form as x, but the ordered subpartition of π associated to each $a_1\bar{x}ya_m$ belongs to $OP_2(\bar{p}-1)$, with some $\bar{p}<m$. Therefore $\varphi(a_1x^{(j)}cya_m)=0$, for every $j=1,...,k$, via the inductive hypothesis. Moreover, $\varphi(a_1x^\circ cya_m)=0$, by Lemma 3.1, since $a_1x^\circ cya_m$ is a simple random variable in (A,φ,ψ,θ).

We conclude by induction.□

Due to Lemma 3.1 and Lemma 3.6, we extend below [15, Lemma 3.7], [16, Lemma 3.4] and [17, Lemma 3.4].

Lemma 3.7 *Let $w=a_1\cdots a_n\in A$, s.t. all $a_j\in A_{i_j}$ are φ,ψ,θ-centered, and the ordered partition π associated to w is a crossing pairing. Then $\varphi(w)=0$.*

Proof. In view of Lemma 3.3 (the weak independence) and Lemma 3.1, it remains to consider: π is irreducible, and w has (under its reduced form) local extrema arising from some interval blocks of π.

For $n=6$, there are involved only the crossing pairings listed below: $\{(1,5)_s,(4,n)_t,(2,3)_3\}$, $\{(1,5)_s,(2,n)_t,(3,4)_3\}$, $\{(1,3)_s,(2,n)_t,(4,5)_3\}\in OP_2(6)$, with $s,t\in\{1,2\}$; and

$\{(2,3)_1,(1,5)_u,(4,n)_v\}$, $\{(3,4)_1,(1,5)_u,(2,n)_v\}$, $\{(4,5)_1,(1,3)_u,(2,n)_v\}\in OP_2(6)$, with $u,v\in\{2,3\}$.

For each of them, the unique local extremum e in the reduced form of w arises as a peak in the first list above from the interval block $(\bullet,\bullet)_3$; but, it arises as a bottom from the interval block $(\bullet,\bullet)_1$ in the former list. The subwords obtained from w by excluding or by centering e, adequately to its type, are both simple random variable in (A,φ,ψ,θ) ; so, $\varphi(w)=0$, always, by Lemma 3.1 .

Let $n>6$, and the statement true for all $p<n$. Then, for $w=a_1\cdots a_n\in A$, the inferences below help to conclude by induction.

Let $a_l a_{l+1} =: e_l \in A_{i_l}$ be, for instance, the left local extremum in the reduced form of w, arising (as a singleton) in w from an interval block $(l,l+1)_{.}\in\pi$, with $i_{l-1}<i_l=i_{l+1}>i_{l+2}$ or $i_{l-1}>i_l=i_{l+1}<i_{l+2}$, if e_l is a peak or a bottom in w, respectively. Then we may express $w=a_1xe_lya_n$ as in Lemma 3.6; where x,y are void or arbitrary products of $a_j\in A_{i_j}$ with $\varphi(a_j)=0=\psi(a_j)=\theta(a_j)$; but, x (as non-void factor in w) has no local extrema arising from interval blocks of π. By the reducing of $x,y,$ and the adequate centering of e_l, we get now $w=b_la_1xya_n+a_1xe_l^{\circ}ya_n$, with $b_l:=\psi(e_l)$ or $b_l:=\theta(e_l)$, if e_l is a peak or a bottom in w, respectively; and $e_l^{\circ}:=e_l-b_l\cdot 1$. The ordered subpartition of π associated to a_1xya_n is crossing and belongs to $OP_2(n-2)$. Hence $\varphi(w)=0$, because a_1xya_n and $a_1xe_l^{\circ}ya_n$ belong to the kernel of φ, by the inductive hypothesis, and Lemma 3.6, respectively.□

If (A,ω) is a quantum probability space, and $x_1,x_2\in A$ are random variables s.t. one of them is ω-centered, then $\omega(x_1x_2)=k_2^{\omega}(x_1,x_2)$; whenever, e.g., k_2^{ω} is the tensor/free/Boolean/monotone joint cumulant (see, e.g., [1, 28]) w.r.t. ω of order two. In the sequel, we may use one of these choices.

In general, the scalars involved below $\bar{k}_\pi(a_1,...,a_n)$, for $\pi\in ONC_2(n)$ and $a_j\in A_{i_j}$, can be described as follows; compare with the c-free case [3, 15], and c-monotone case ;see, e.g. [17] .

1) If π has a single block, then that is an outer block of π, and $\bar{k}_\pi(a_1,a_2):=k_2^{\varphi}(a_1,a_2)$.

2) If $\pi=\sigma\coprod\rho$, with $\sigma\in ONC_2(i)$ and $\rho\in ONC_2(\{i+1,...,n\})$, then

$$\bar{k}_\pi(a_1,...,a_n):=\bar{k}_\sigma(a_1,...,a_i)\cdot\bar{k}_\rho(a_{i+1},...,a_n).$$

3)If π consists of the block $(1,n)_{.}$, and the subpartition $\sigma=\pi\cap\{2,...,n-1\}\in ONC_2(\{2,...,n-1\})$, then

$\bar{k}_\pi(a_1,...,a_n):=k_2^{\varphi}(a_1,a_n)k_\sigma(a_2,...,a_{n-1})$; where the scalars $k_\sigma(a_2,...,a_{n-1})$ are described in the following way, with $n\geq 3$. More generally, for a subpartition σ of $\pi\in ONC_2(n)$, i.e., $\sigma\in ONC_2(S)$ and $S\subset\{i_1,...,i_n\}$, say $S:=\{\iota_1,...,\iota_s\}$, define $k_\sigma(x_1,...,x_s)$, for $x_j\in A_{\iota_j}$ as follows.

i) If σ has a single block, then that is an inner block $(\cdot,\cdot)_l$ of π, and let $(\cdot,\cdot)_k$ be its nearst cover in π .Then

$k_\sigma(x_1,x_s):=k_2^{\psi}(x_1,x_s)$ if $l>k$, but $k_\sigma(x_1,x_s):=k_2^{\theta}(x_1,x_s)$ if $l<k$.

ii)If $\sigma=\rho\coprod\tau$, with $\rho\in ONC_2(S_1)$, and $\tau\in ONC_2(S_2)$, then

$k_\sigma(x_1,...,x_s):=k_\rho(x_i,i\in S_1)\cdot k_\tau(x_i,i\in S_2)$.□

Some computations of the scalars $\bar{k}_\pi(a_1,...,a_n)$, for $\pi\in ONC_2(n)$ are presented below.

Examples 3.8 1) For $n=8$, let $\pi_1=((2,5)_1,(3,4)_2(1,8)_3,(6,7)_4)$, and $\pi_2=((2,3)_1,(1,8)_2,(5,6)_3,(4,7)_4)$.

By considering $\eta:=((3,4)_2)$ and $\tau:=((6,7)_4)$ as ordered subpairings of π_1, we get $k_\eta(a_3,a_4)=k_2^\psi(a_3,a_4)=:b_2$, and $k_\tau(a_6,a_7)=k_2^\psi(a_6,a_7)=:b_1$, because the nearest cover in π_1 of $(3,4)_2$ is $(2,5)_1$ and of $(6,7)_4$ is $(1,n)_3$. Thus, for $\rho:=((2,5)_1,(3,4)_2)$ we get by definition $k_\rho(a_2,...,a_5)=k_2^\theta(a_2,a_5)b_2$, the nearest cover in π_1 of $(2,5)_1$ being $(1,n)_3$. Then, for the ordered subpairing of $\pi_1=((1,n)_3)\coprod\sigma$ defined by $\sigma:=\rho\coprod\tau$, the definition of k_σ implies

$$k_\sigma(a_2,...,a_7)=k_\rho(a_2,...,a_5)k_\tau(a_6,a_7)=k_2^\theta(a_2,a_5)b_2b_1.$$

Hence $\overline{k}_{\pi_1}(a_1,...,a_n)=k_2^\varphi(a_1,a_n)k_\sigma(a_2,...,a_{n-1})=k_2^\varphi(a_1,a_n)k_2^\theta(a_2,a_5)b_2b_1$.

In the second case, consider $\rho:=((2,3)_1)$, $\omega:=((4,7)_4)$, $\eta:=((5,6)_3)$ as ordered subpairings of π_2, to get $k_\rho(a_2,a_3)=k_2^\theta(a_2,a_3)=:b_2$ and $k_\eta(a_5,a_6)=k_2^\theta(a_5,a_6)=:b_1$, because the nearest cover in π_2 of $(2,3)_1$ is $(1,n)_2$ and of $(5,6)_3$ is $(4,7)_4$. Then for $\tau:=\omega\coprod\eta$, the definition of k_τ implies $k_\tau(a_4,...,a_7)=k_2^\psi(a_4,a_7)b_1$, the nearest cover in π_2 of ω being $(1,n)_2$. Thus, for the ordered subpairing of $\pi_2=((1,n)_2)\coprod\sigma$ defined by $\sigma:=\rho\coprod\tau$, the definition of k_σ implies now

$$k_\sigma(a_2,...,a_7)=k_\rho(a_2,a_3)k_\tau(a_4,...,a_7)=b_2k_2^\psi(a_4,a_7)b_1.$$

Hence $\overline{k}_{\pi_2}(a_1,...,a_n)=k_2^\varphi(a_1,a_n)k_\sigma(a_2,...,a_{n-1})=k_2^\varphi(a_1,a_n)\,b_2k_2^\psi(a_4,a_7)b_1$.

2) For $n=10$, let $\pi_3=((2,5)_1,(3,4)_2(1,n)_3,(7,8)_4,(6,9)_5)$. Denote in this case $\omega':=((2,5)_1)$, $\omega:=((3,4)_2)$, $\eta':=((6,9)_5)$ and $\eta:=((7,8)_4)$ as ordered subpairings of π_3, to get $k_\omega(a_3,a_4)=k_2^\psi(a_3,a_4)=:b_2$, and $k_\eta(a_7,a_8)=k_2^\theta(a_7,a_8)=:b_1$; because the nearest cover in π_3 of the involved inner block belonging to ω, η is $(2,5)_1$, respectively, $(6,9)_5$. Then for $\rho:=\omega'\coprod\omega$ and $\tau:=\eta'\coprod\eta$, it follows $k_\rho(a_2,...,a_5)=k_2^\theta(a_2,a_5)b_2$ and $k_\tau(a_6,...,a_9)=k_2^\psi(a_6,a_9)b_1$; the nearest cover in π_3 of the blocks from ω' and η' being $(1,n)_3$. These imply, for the ordered subpairing of $\pi_3=((1,n)_3)\coprod\sigma$ defined by $\sigma:=\rho\coprod\tau$,

$k_\sigma(a_2,...,a_9)=k_\rho(a_2,...,a_5)k_\tau(a_6,...,a_9)=k_2^\theta(a_2,a_5)b_2\,k_2^\psi(a_6,a_9)b_1$; and finally

$$\overline{k}_{\pi_3}(a_1,...,a_n)=k_2^\varphi(a_1,a_n)k_\sigma(a_2,...,a_{n-1})=k_2^\varphi(a_1,\,a_n)\,k_2^\theta(a_2,a_5)b_2\,k_2^\psi(a_6,a_9)b_1.\square$$

We illustrate the next lemma by the following classes of partitions in $ONC_2(n)$ associated to $w=a_1\cdots a_n\in A$. Compare with [16, Ex. 3.5] and [17, Ex. 3.7].

Examples 3.9 1) For $n=4$, let $\pi_1=((2,3)_1,(1,4)_2)$ and $\pi_2=((1,4)_1,(2,3)_2)$, which are anti-monotone and, respectively, monotone. In both cases, the unique local extremum $e:=a_2a_3$ arises from the block $(2,3)_\bullet$: as a bottom for π_1, and a peak for π_2; and the nearest cover of this block is $(1,4)_\bullet$, in each case. The subword obtained from w by centering e, adequately to its type, is a simple random variable in (A,φ,ψ,θ); so, it belongs to the kernel of φ, by Lemma 3.1. Thus, $\varphi(w)$ equals the value of φ on the subword obtained from w by excluding e, adequately to its type.

Hence, $\varphi(w)=k_2^\varphi(a_1,a_4)b$, with $b:=k_2^\theta(a_2,a_3)$, for π_1, but $b:=k_2^\psi(a_2,a_3)$ for π_2; giving, by definition, $\overline{k}_{\pi_i}(a_1,...,a_4)$, respectively.

2) For $n=6$, let π be any of the ordered non-crossing pairings listed below:

i) $((1,6)_1,(2,5)_2,(3,4)_3)\in M_2(6)$, $((3,4)_1,(2,5)_2,(1,6)_3)\in AM_2(6)$, and

$\{(2,5)_1,(3,4)_s,(1,6)_t\}$, $\{(1,6)_u,(3,4)_v,(2,5)_3\} \in ONC_2(6)\setminus(M_2(6)\cup AM_2(6))$, with $s,t\in\{2,3\}$, respectively, $u,v\in\{1,2\}$;

ii) $\{(2,3)_s,(4,5)_t,(1,n)_3\}\in AM_2(6)$, with $s,t\in\{1,2\}$, and $((2,3)_1,(1,n)_2,(4,5)_3)$ $\in ONC_2(6)\setminus(M_2(6)\cup AM_2(6))$.

For each partition from the first list i), the unique non-centered local extremum $e := a_3a_4$ emerges from the block $(3,4)_{\cdot}$ as a peak in the monotone case or the first non-(anti-)monotone cases, but as a bottom, otherwise. Remark a_2 and a_5 are local extrema in the non-(anti-)monotone cases, but they are already centered adequately to their type. In all cases, the subword obtained from w by centering e, adequately to its type, is a simple random variable in (A,φ,ψ,θ); so, it is in the kernel of φ by Lemma 3.1. Therefore, $\varphi(w)$ equals the value of φ on the subword obtained from w by excluding e, adequately to its type. The computation reduces to the above examples for $n=4$, giving, with $b := k_2^{\psi}(a_3,a_4)$ when e is a peak, but with $b := k_2^{\theta}(a_3,a_4)$, when e is a bottom,

$\varphi(w) = k_2^{\varphi}(a_1,a_n)k_{\rho}(a_2,a_5)b$; denoting $\rho := ((2,5)_{\cdot})$ as ordered subpairing of $\{(1,n)_{\cdot},(2,5)_{\cdot}\}$.

By considering $\tau := ((3,4)_{\cdot})$ as ordered subpairing of π, we get $k_{\tau}(a_3,a_4)=b$ always, because the nearest cover in π of the block $(3,4)_{\cdot}$ is the block $(2,5)_{\cdot}$. Consequently, the definition of k_{σ} and $\overline{k}_{\pi}$, where $\sigma := \rho \coprod \tau$ as ordered subpairing of $\pi = ((1,n)_{\cdot})\coprod\sigma$, imply

$k_{\sigma}(a_2,...,a_5) = k_{\rho}(a_2,a_5)k_{\tau}(a_3,a_4) = k_{\rho}(a_2,a_5)b$, and finally

$\varphi(w) = k_2^{\varphi}(a_1,a_n)k_{\sigma}(a_2,...,a_5) = \overline{k}_{\pi}(a_1,...,a_n)$ always.

Explicitly, $\varphi(w)$ equals:

$k_2^{\varphi}(a_1,\, a_n)k_2^{\psi}(a_2,a_5)k_2^{\psi}(a_3,a_4)$, in the monotone case;

$k_2^{\varphi}(a_1,\, a_n)k_2^{\theta}(a_2,a_5)k_2^{\psi}(a_3,a_4)$ and $k_2^{\varphi}(a_1,\, a_n)k_2^{\psi}(a_2,a_5)k_2^{\theta}(a_3,a_4)$, respectively, in the non-(anti-) monotone cases;

$k_2^{\varphi}(a_1,\, a_n)k_2^{\theta}(a_2,a_5)k_2^{\theta}(a_3,a_4)$, in the anti-monotone case.

Concerning the anti-monotone partitions from the second list ii) above, remark the unique non-centered local extremum e arises as a bottom from $(2,3)_{\cdot}$, respectively $(4,5)_{\cdot}$ In both cases, the subword obtained from w by centering e, adequately to its type, is a simple random variable in (A,φ,ψ,θ); so, it belongs to the kernel of φ by Lemma 3.1. Therefore, once again $\varphi(w)$ equals the value of φ on the subword obtained from w by excluding e, adequately to its type. According to the above anti-monotone example for $n=4$, we deduce

$\varphi(w) = k_2^{\varphi}(a_1,\, a_n)bk_2^{\theta}(a_4,a_5)$ with $b := k_2^{\theta}(a_2,a_3)$, respectively

$\varphi(w) = k_2^{\varphi}(a_1,\, a_n)k_2^{\theta}(a_2,a_3)\overline{b}$ with $\overline{b} := k_2^{\theta}(a_4,a_5)$.

By considering $\rho := ((2,3)_{\cdot})$ and $\tau := ((4,5)_{\cdot})$ as ordered subpairings of π, note that $k_{\rho}(a_2,a_3)=b$ and $k_{\tau}(a_4,a_5)=\overline{b}$, because $(1,n)_{\cdot}$ is the nearest cover in π of both blocks involved.

The definition of k_{σ} and $\overline{k}_{\pi}$, where $\sigma := \rho\coprod\tau$ as ordered subpairing of $\pi = ((1,n)_{\cdot})\coprod\sigma$, imply this time

$k_{\sigma}(a_2,...,a_5) = k_{\rho}(a_2,a_3)\; k_{\tau}(a_4,a_5) = b\overline{b}$, but finally

$\varphi(w) = k_2^{\varphi}(a_1,\, a_n)k_2^{\theta}(a_2,a_3)k_2^{\theta}(a_4,a_5) = \overline{k}_{\pi}(a_1,...,a_n)$.

For the non-(anti-)monotone pairing π remaining in the list ii), the local extrema e_2, e_1 arise from the interval blocks $(2,3)_1$ and $(4,5)_3$ as a bottom and, respectively, a peak. By centering each of e_2, e_1 adequately to its type $e_j = b_j \cdot 1 + e_j^\circ$, where the scalars $b_j \in \mathbb{C}$ are given by $b_2 := \theta(e_2)$, and $b_1 := \psi(e_1)$, and $e_j^\circ := e_j - b_j \cdot 1$, we express $x := e_2 e_1 = x_\circ + x_1 + x_2 + x^\circ$, with $x_\circ := b_1 b_2$, $x_1 := b_1 e_2^\circ$, $x_2 := b_2 e_1^\circ$, and $x^\circ := e_2^\circ e_1^\circ$, and note the random variables $a_1 x_1 a_n$, $a_1 x_2 a_n$ and $a_1 x^\circ a_n$ are simple in $(A, \varphi, \psi, \theta)$, being in $\ker\varphi$ by Lemma 3.1. Therefore, $\varphi(w) = \varphi(a_1 x_\circ a_n) = k_2^\varphi(a_1, a_n) b_2 b_1$. By regarding $\rho := ((2,3)_1)$ and $\tau := ((4,5)_3)$ as ordered subpairings of π, note that $k_\rho(a_2, a_3) = b_2$ and $k_\tau(a_4, a_5) = b_1$, because $(1,n)_\cdot$ is the nearest cover in π of both blocks involved.

The definition of k_σ and $\overline{k}_\pi$, where $\sigma := \rho \coprod \tau$ as ordered subpairing of $\pi = ((1,n)_\cdot) \coprod \sigma$, imply in this case

$k_\sigma(a_2, ..., a_5) = k_\rho(a_2, a_3)\, k_\tau(a_4, a_5) = b_2\, b_1$ and finally

$\varphi(w) = k_2^\varphi(a_1, a_n) k_2^\theta(a_2, a_3) k_2^\psi(a_4, a_5) = \overline{k}_\pi(a_1, ..., a_n)$.

3) For $n = 8$, and $n = 10$, let return to the ordered non-crossing pairings from the Examples 3.8, which are non-(anti-)monotone: $\pi_1 = ((2,5)_1, (3,4)_2 (1,8)_3, (6,7)_4)$, $\pi_2 = ((2,3)_1, (1,8)_2, (5,6)_3, (4,7)_4)$, and, respectively, $\pi_3 = ((2,5)_1, (3,4)_2 (1,n)_3, (7,8)_4, (6,9)_5)$.

For π_1, the non-centered local extrema e_2, e_1 arise from the interval blocks $(3,4)_2$ and $(6,7)_4$ as peaks. By centering e_2, e_1 adequately to their type $e_j = b_j \cdot 1 + e_j^\circ$, where the scalars $b_j \in \mathbb{C}$ are given by $b_2 := \psi(e_2)$, and $b_1 := \psi(e_1)$, and $e_j^\circ := e_j - b_j \cdot 1$, we express $x := u e_2 v e_1 = x_\circ + x_1 + x_2 + x^\circ$, with $x_\circ := b_1 b_2 uv$, $x_1 := b_1 u e_2^\circ v$, $x_2 := b_2 u v e_1^\circ$, $x^\circ := u e_2^\circ v e_1^\circ$, $u = a_2$ and $v = a_5$ and note the random variables $a_1 x_1 a_n$ and $a_1 x^\circ a_n$ are simple in $(A, \varphi, \psi, \theta)$, being in $\ker\varphi$ by Lemma 3.1; moreover, $\varphi(a_1 x_2 a_n) = 0$ due to an example before, for $n = 6$, from the list ii). By the anti-monotone example for $n = 4$ before, we get $\varphi(w) = \varphi(a_1 x_\circ a_n) = k_2^\varphi(a_1, a_n) k_2^\theta(a_2, a_5) b_2 b_1 = \overline{k}_{\pi_1}(a_1, ..., a_n)$ according to Examples 3.8.

For π_2, the non-centered local extrema e_2, e_1 arise from the interval blocks $(2,3)_1$ and $(5,6)_3$ as bottoms. By centering e_2, e_1 adequately to their type $e_j = b_j \cdot 1 + e_j^\circ$, where $b_j \in \mathbb{C}$ are given by $b_2 := \theta(e_2)$, and $b_1 := \theta(e_1)$, and $e_j^\circ := e_j - b_j \cdot 1$, we express $x := e_2 v_2 e_1 v_1 = x_\circ + x_1 + x_2 + x^\circ$, with $x_\circ := b_1 b_2 v_2 v_1$, $x_1 := b_1 e_2^\circ v_2 v_1$, $x_2 := b_2 v_2 e_1^\circ v_1$, $x^\circ := e_2^\circ v_2 e_1^\circ v_1$, $v_2 = a_4$ and $v_1 = a_7$ and note the random variables $a_1 x_2 a_n$ and $a_1 x^\circ a_n$ are simple in $(A, \varphi, \psi, \theta)$, belonging to $\ker\varphi$ by Lemma 3.1; moreover, $\varphi(a_1 x_1 a_n) = 0$ due to the same example before, for $n = 6$, from the list ii). But now, by the monotone example for $n = 4$ before, $\varphi(w) = \varphi(a_1 x_\circ a_n) = k_2^\varphi(a_1, a_n) b_2 k_2^\psi(a_4, a_7) b_1 = \overline{k}_{\pi_2}(a_1, ..., a_n)$ according to Examples 3.8.

For π_3, the non-centered local extrema e_2, e_1 arise from the interval blocks $(3,4)_2$ and $(7,8)_4$ as a peak and, respectively, a bottom. By centering each of e_2, e_1 adequately to its type $e_j = b_j \cdot 1 + e_j^\circ$, where $b_j \in \mathbb{C}$ are given by $b_2 := \psi(e_2)$, and $b_1 := \theta(e_1)$, and $e_j^\circ := e_j - b_j \cdot 1$, we develop $x := u e_2 v_2 e_1 v_1 = x_\circ + x_1 + x_2 + x^\circ$, with $x_\circ := b_1 b_2 u v_2 v_1$, $x_1 := b_1 u e_2^\circ v_2 v_1$, $x_2 := b_2 u v_2 e_1^\circ v_1$, $x^\circ := u e_2^\circ v_2 e_1^\circ v_1$, $u = a_2, v_2 = a_5 a_6$ and $v_1 = a_9$ and note the random variable $a_1 x^\circ a_n$ is simple in $(A, \varphi, \psi, \theta)$, being in $\ker\varphi$ by Lemma 3.1. Moreover, $a_1 x_1 a_n$, $a_1 x_2 a_n$ are not simple random variables, but these belong also to $\ker\varphi$, by the previous examples for $n=6$ concerning π_1 and, respectively, π_2. Therefore, by the same example before, for $n=6$, from the list ii), we get

$\varphi(w)=\varphi(a_1x_\circ a_n)=k_2^{\varphi}(a_1,a_n)\ k_2^{\theta}(a_2,a_5)b_2k_2^{\psi}(a_6,a_9)b_1=\overline{k}_{\pi_3}(a_1,...,a_n)$ according to Examples 3.8.□

Due to Lemma 3.1 again, and Lemma 3.6, we get the adequate extension of [15, Lemma 3.8],[16, Lemma 3.6] and [17, Lemma 3.8] by a combined argument; via Lemma 3.3.

Lemma 3.10 *Let* $w=a_1\cdots a_n\in A$, *s.t. all* $a_j\in A_{i_j}$ *are* φ,ψ,θ*-centered and the ordered partition* π *associated to* w *is a non-crossing pairing. Then* $\varphi(w)=\overline{k}_\pi(a_1,...,a_n)$.

Proof. In view of Lemma 3.3 (the weak independence), and Lemma 3.1, we may consider $(1,n)_\cdot\in\pi$, and w has (under its reduced form) local extrema arising from some interval blocks of π. Note that, for the pair of blocks $P_k\prec P_l$, where P_k is the nearst cover of P_l, in π, involving such an interval block giving the left (right) local extremum e in w, the colouring is order-reflecting: $k<l$, if e is a peak; but, it is order-reversing: $k>l$, if e is a bottom.

For $n=4$, see Examples 3.9. For $n=6$, there are only the following pairings π, besides the partitions in Examples 3.9:

$\{(1,n)_1,(2,3)_s,(4,5)_t\}\in ONC_2(6)$, $s,t\in\{2,3\}$, and $((4,5)_1,(1,n)_2,(2,3)_3)\in ONC_2(6)\setminus(M_2(6)\cup AM_2(6))$.

For the monotone pairings above, the single non-centered local extremum e arises from the interval block $(\cdot,\cdot)_3$ as a peak. In both cases, the subword obtained from w by centering e, adequately to its type, is a simple random variable in (A,φ,ψ,θ); so, it is in the kernel of φ by Lemma 3.1. Therefore, once again $\varphi(w)$ equals the value of φ on the subword obtained from w by excluding e, adequately to its type. The computation reduces to the monotone example for $n=4$, giving in both cases

$\varphi(w)=k_2^{\varphi}(a_1,a_n)k_2^{\psi}(a_2,a_3)k_2^{\psi}(a_4,a_5)$. By considering $\rho:=((2,3)_\cdot)$ and $\tau:=((4,5)_\cdot)$ as ordered subpairings of the monotone subpartition $\sigma:=\{(2,3)_\cdot,(4,5)_\cdot\}\in ONC_2(4)$ of π, we get $k_\rho(a_2,a_3)=k_2^{\psi}(a_2,a_3)$ and $k_\tau(a_4,a_5)=k_2^{\psi}(a_4,a_5)$, because $(1,n)_1$ is the nearst cover in π of both blocks $(2,3)_\cdot$ and $(4,5)_\cdot$. Hence $\varphi(w)=k_2^{\varphi}(a_1,a_n)k_\sigma(a_2,...,a_5)=\overline{k}_\pi(a_1,...,a_n)$, in both cases, by definition of $\overline{k}_\pi$.

For the non-(anti-)monotone pairing above, the local extrema e_2,e_1 arise from the interval blocks $(\cdot,\cdot)_3$ and $(\cdot,\cdot)_1$ as a peak and , respectively, a bottom. By centering each of e_2,e_1 adequately to its type $e_j=b_j\cdot 1+e_j^\circ$, where $b_j\in\mathbb{C}$ are given by $b_2:=\psi(e_2)$, and $b_1:=\theta(e_1)$, and $e_j^\circ:=e_j-b_j\cdot 1$, we develop $x:=e_2e_1=x_\circ+x_1+x_2+x^\circ$, with $x_\circ:=b_1b_2$, $x_1:=b_1e_2^\circ$, $x_2:=b_2e_1^\circ$, and $x^\circ:=e_2^\circ e_1^\circ$, and note the random variables $a_1x_1a_n$, $a_1x_2a_n$ and $a_1x^\circ a_n$ are simple in (A,φ,ψ,θ), belonging to $\ker\varphi$ by Lemma 3.1. Hence $\varphi(w)$ equals $\varphi(a_1x_\circ a_n)=k_2^{\varphi}(a_1,a_n)b_2b_1=k_2^{\varphi}(a_1,a_n)k_2^{\psi}(a_2,a_3)k_2^{\theta}(a_4,a_5)=\overline{k}_\pi(a_1,...,a_n)$, by definition of $\overline{k}_\pi$; because $(1,n)_2$ is the nearst cover in π of both blocks $(2,3)_3$ and $(4,5)_1$.

Let $n>6$. Suppose the assertion true for all $p<n$. To conclude by induction, remark the next facts.

Let $a_ra_{r+1}=:e_r\in A_{i_r}$ be, for instance, the right local extremum in the reduced form of w, arising (as a singleton) in w from an interval block $(r,r+1)_l\in\pi$, with $i_{r-1}<i_r=i_{r+1}>i_{r+2}$ or $i_{r-1}>i_r=i_{r+1}<i_{r+2}$, if e_r is a peak, or a bottom in w, respectively. Denote by k the order of the nearst cover of this block in π. Let $\eta:=((r,r+1)_l)$ as ordered subpartition of π. Then we may express $w=a_1xe_rya_n$ as in Lemma 3.6; where x,y are void or arbitrary products of $a_j\in A_{i_j}$ with $\varphi(a_j)=0=\psi(a_j)=\theta(a_j)$; but, y (as

non-void factor in w) has no local extrema arising from interval blocks of π. After the reducing of x and y, and the adequate centering of e_r, we get now $w = b_r a_1 x y a_n + a_1 x e_r^\circ y a_n$, with the scalars $b_r := \psi(e_r)$ or $b_r := \theta(e_r)$, if e_r is a peak, or a bottom in w, respectively; and $e_r^\circ := e_r - b_r \cdot 1$. Now, the ordered subpartition of π associated to $a_1 x y a_n$ belongs to $ONC_2(n-2)$. Thus, $\varphi(a_1 x e_r^\circ y a_n) = 0$, by Lemma 3.6, again. Then, by definition, $k_\eta(a_r, a_{r+1}) = k_2^\psi(a_r, a_{r+1})$, if e_r is a peak (when $l > k$); but $k_\eta(a_r, a_{r+1}) = k_2^\theta(a_r, a_{r+1})$, if e_r is a bottom (when $l < k$). Thus, $k_\eta(a_r, a_{r+1}) = b_r$ always.

We may proceed as in [15, Lemma 3.8], [16, Lemma 3.6] and [17, Lemma 3.8]. Let $\rho \in ONC_2(\{2,...,n-1\} \setminus \{r, r+1\})$ be the ordered sub-partition of π associated to xy and $\{(1,n).\} \coprod \rho =: \sigma \in ONC_2(\{1,...,n\} \setminus \{r, r+1\})$ be the ordered sub-partition of π associated to $a_1 x y a_n$. The induction hypothesis and the definition of $\bar{k}_\sigma$ imply

$$\varphi(a_1 x y a_n) = \bar{k}_\sigma(a_1, a_2, ..., a_{r-1}, a_{r+2}, ..., a_{n-1}, a_n) = k_2^\varphi(a_1, a_n) k_\rho(a_2, ..., a_{r-1}, a_{r+2}, ..., a_{n-1}).$$

But $k_\rho(a_2, ..., a_{r-1}, a_{r+2}, ..., a_{n-1}) b_r = k_\rho(a_2, ..., a_{r-1}, a_{r+2}, ..., a_{n-1}) k_\eta(a_r, a_{r+1}) =$

$= k_\tau(a_2, ..., a_{r-1}, a_r, a_{r+1}, a_{r+2}, ..., a_{n-1})$, by definition of k_τ; denoting $\tau := \rho \coprod \eta$ as ordered subpartition of π.

And note $\pi = ((1,n).) \coprod \tau$; hence $\varphi(w) = k_2^\varphi(a_1, a_n) k_\tau(a_2, ..., a_{n-1}) = \bar{k}_\pi(a_1, ..., a_n)$ by definition of $\bar{k}_\pi$, too.□

Remarks 3.11 *Moreover, for the random variable* w *in the previous statement, Lemma 3.6 implies* $\varphi(w) = \varphi(w_\circ)$, *where* $w_\circ$ *is the subword obtained from* w *by excluding, adequately to their types, all the local extrema not centered adequately (when they exist) that arise, in the reduced form of* w *or in the reduced form of any subword* w' *obtained from* w *by excluding, adequately to its type, any local extremum not centered adequately.*

Similar identities describe the expectation $\varphi(w)$ *concerning the random variable* w *in* [15, Lemma 3.8], [16, Lemma 3.6] *and* [17, Lemma 3.8]. *Namely,* $w_\circ$ *is the subword obtained from* w *by excluding: w.r.t.* ψ, *for the c-free and c- monotone cases, all non* ψ*-centered singletons, respectively, all internal peaks (when they exist) arising, in the reduced form of* w, *or in the reduced form of any subword* w' *obtained from* w *by excluding, w.r.t.* ψ, any *singleton, respectively, any internal peak non* ψ*-centered*; *for the monotone case, by excluding all non-centered internal peaks, in the same conditions, etc.*□

This final lemma extends [17, Lemma 3.9] to a family of φ, ψ, θ-indentedly independent sets of random variables.

Lemma 3.12 *Let* $a_i = (a_i^s)_{s \in S}, i \in I$ *be random vectors in a probability space* $(A, \varphi, \psi, \theta)$, *such that* $\{a_i^s, s \in S\} \subset A, i \in I$ *are* φ, ψ, θ*-indentedly independent sets of random variables in* $(A, \varphi, \psi, \theta)$, *and* $a_i = (a_i^s)_{s \in S}, i \in I$ *have the same joint distribution w.r.t.* φ, ψ, θ. *Then the joint moments of* $(a_i)_{i \in I}$ *w.r.t.* φ *are invariant under order-preserving injective maps; i.e., for all* n, *all* $s_1, ..., s_n \in S$, *all* $i_1, ..., i_n \in I$ *and all order-preserving injection* $\sigma : \{i_1, ..., i_n\} \to I$, *it holds* $\varphi(a_{i_1}^{s_1} ... a_{i_n}^{s_n}) = \varphi(a_{\sigma(i_1)}^{s_1} ... a_{\sigma(i_n)}^{s_n})$.

Proof. Since $a_i, i \in I$ are identically distributed w.r.t. φ, we get the statement if all i_l are equal. Otherwise, assume the statement valid for any $p < n$.

Let consider $l \mapsto i_l$ has no intervals.

If $l \mapsto i_l$ has no local extrema, then $l \mapsto \sigma(i_l)$ has no local extrema, too, these both maps are strictly monotone and the statement for n follows via Lemma 3.1 and the same non-inductive hypothesis cited above for $a_i, i \in I$.

Alternatively, denote $a_{i_l}^{s_l} =: c_l, a_{\sigma(i_l)}^{s_l} =: d_l$, for $l = 1,..,n$, and the random variables from both sides by $w := c_1 \cdots c_n$ and $w' := d_1 \cdots d_n$. If $l \mapsto i_l$ has i_p as local extremum, then $l \mapsto \sigma(i_l)$ has $\sigma(i_p)$ as local extremum of the same type. In other words, if c_p is a local extremum in w, then d_p is a local extremum of the same type and on the same place $p \equiv \sigma(p)$ in w'.

Suppose w has exactly k local extrema $e_1,...,e_k$; thus, w' has exactly k local extrema $f_1,...,f_k$ of the same type and on the same places, respectively. We may express $w = c_1 x c_n$, $w' = d_1 y d_n$ with $x = u e_k v_k \cdots e_1 v_1$, $y = \overline{u} f_k \overline{v}_k \cdots f_1 \overline{v}_1$ with u, v_r and $\overline{u}, \overline{v}_r$ being void or arbitrary products of c_l and, respectively, d_l ; $l = 1,...,n$. But u, v_r and $\overline{u}, \overline{v}_r$ as non-void factors in x or y have no local extrema.

By centering each pair of local extrema e_r, f_r adequately to their type, and c_n, d_n w.r.t. φ, note that $c_n = b_n \cdot 1 + c_n^\circ$, then $d_n = b_n \cdot 1 + d_n^\circ$ with $b_n := \varphi(c_n) = \varphi(d_n)$, and $e_r = b_r \cdot 1 + e_r^\circ$, $f_r = b_r \cdot 1 + f_r^\circ$ with the same scalars $b_r \in \mathbb{C}$; because $a_i, i \in I$ are identically distributed w.r.t. φ, ψ, θ.

Consequently, by developing x and y in the pre-centered form $x = \sum_{l=1}^{k} b_l x^{(l)} + x^\circ$ and $y = \sum_{l=1}^{k} b_l y^{(l)} + y^\circ$ (with corresponding $x^{(l)}, x^\circ$ and, respectively, $y^{(l)}, y^\circ$ defined as in Preliminaries), we get $w = b_n c_1 x + \sum_{l=1}^{k} b_l c_1 x^{(l)} c_n^\circ + c_1 x^\circ c_n^\circ$, and $w' = b_n d_1 y + \sum_{l=1}^{k} b_l d_1 y^{(l)} d_n^\circ + d_1 y^\circ d_n^\circ$.

All $x^{(l)}$ and $y^{(l)}$, $l = 1,...,k$ express as algebraic sums (with ± 1 as coefficients) of random variables $\overline{x}$, respectively $\overline{y}$ having the same generic form as x, and, respectively, y. But the length of $c_1 \overline{x} c_n^\circ$ and $d_1 \overline{y} d_n^\circ$ is obviously inferior to n. By the inductive hypothesis, φ takes equal values on $c_1 x$ and $d_1 y$ or on all the terms $c_1 \overline{x} c_n^\circ$ and $d_1 \overline{y} d_n^\circ$ of the same length, from each pair $c_1 x^{(l)} c_n^\circ$ and $d_1 y^{(l)} d_n^\circ$ (l being fixed).

Hence the statement for n is valid; because the terms of maximum length $c_1 x^\circ c_n^\circ$ and $d_1 y^\circ d_n^\circ$ are in the kernel of φ by Lemma 3.1 as simple random variables in $(A, \varphi, \psi, \theta)$.

When $l \mapsto i_l$ has intervals, the statement for n follows by the same reasoning as above, after a reducing of the random variables from both sides. Therefore, the statement being clear for $n \leq 3$, we conclude by induction.□

4. INDENTED GAUSSIAN FAMILY AND MULTIVARIATE CLT

Let I be an arbitrary index set. We remind a scalar matrix $q = \{q_{ij}\}_{i,j \in I}$ is positive if and only if $\sum_{k,l=1}^{n} q_{i_k, i_l} \overline{\lambda}_k \lambda_l \geq 0$, for all n, all $i_1,...,i_n \in I$, and all $\lambda_1,...,\lambda_n \in \mathbb{C}$.

The following definition is inspired from [1, 3, 4, 6, 8, 9, 12-17, 28].

Definition 4.1 *Let* $q=\{q_{ij}\}_{i,j\in I}$, $r=\{r_{ij}\}_{i,j\in I}$ *and* $s=\{s_{ij}\}_{i,j\in I}$ *be (positive) scalar matrices. Let* (A,φ) *be a quantum (*-) probability space. A family of (selfadjoint) random variables* $g=(g_i)_{i\in I}$ *in this is called a centered indented Gaussian family of covariances* q, r *and* s*, if its distribution is of the following form, for all* $j\in\mathbb{N}$ *and all* $i_1,\dots,i_j\in I$ *:*

$$\varphi(g_{i_1}\dots g_{i_j})=\sum_{\pi\in ONC_2(j)}\tfrac{1}{|\pi|!}\overline{k}_\pi(g_{i_1},\dots,g_{i_j})\text{; where }\overline{k}_\pi(g_{i_1},\dots,g_{i_j}):=\prod_{(k,l)\in O(\pi)}q_{i_ki_l}\prod_{(k,l)\in I_1(\pi)}r_{i_ki_l}\prod_{(k,l)\in I_2(\pi)}s_{i_ki_l}\,.\ \square$$

Remarks 4.2 1) *If* $q=r$*, we get an ordered-free Gaussian family of covariances* r *and* s.

2) *If* $s=r$*, we get a c-free Gaussian family of covariances* q *and* r*, involving the non-crossing pairings* $NC_2(j)$ *(see, e.g.,* [3, 15]*);in particular, when* $s=r=q$*, we obtain a free Gaussian (: semicircular) family of covariance* q *(see, e.g.,* [28]).

3) *If* $s=0$*, we recover the notion of c-monotone Gaussian family of covariances* q *and* r*, involving the monotone pairings* $M_2(j)$ [17, Def 4.1]*;in particular, when* $s=0$ *and* $r=q$*, we get a monotone Gaussian family of covariance* q *(see, e.g.,* [9, 16]).

4) *If* $r=0$*, we recover the notion of c-anti-monotone Gaussian family of covariances* q *and* s*, involving the anti-monotone pairings* $AM_2(j)$ [17]*;in particular, when* $r=0$ *and* $s=q$ *, we get an anti- monotone Gaussian family of covariance* q (*see, e.g.,* [9, 16]).

5) *If* $s=r=0$*, we obtain a Boolean Gaussian (:Bernoulli) family of covariance* q*, involving the interval pairings* $I_2(j)$ *(see, e.g.,* [15, 17]) *; an empty product being equal to 1 by convention.* $\square$

Theorem 4.3 *Let* (A,φ,ψ,θ) *be a unital quantum (*-)probability space, and* $\{X_r^i, i\in I\}\subset A$, $r\in\mathbb{N}$ *be a sequence of* φ,ψ,θ*-indentedly independent sets of (selfadjoint) random variables in this, s.t.* $X_r=(X_r^i)_{i\in I}$ *has the same joint distribution for all* $r\in\mathbb{N}$ *, and all variables are centered, both w.r.t.* φ,ψ,θ*. Consider, for every* $N\geq 1$*, the sums* $S_N^i:=\frac{1}{\sqrt{N}}\sum_{r=1}^{N}X_r^i\in A$*, and* $S_N:=(S_N^i)_{i\in I}$ *as random vector in* (A,φ)*. Denote the covariances of the variables w.r.t.* φ,ψ,θ *by* $q=\{q_{ij}\}_{i,j\in I}$, $r=\{r_{ij}\}_{i,j\in I}$ *and* $s=\{s_{ij}\}_{i,j\in I}$*; i.e.,* $q_{ij}:=\varphi(X_r^iX_r^j)$, $r_{ij}:=\psi(X_r^iX_r^j)$ *and* $s_{ij}:=\theta(X_r^iX_r^j)$*. Then* $S_N\xrightarrow{distr}g$*; where* $g=(g_i)_{i\in I}$ *is a centered indented Gaussian family of (positive) covariances* q, r *and* s.

Proof. Since all X_r have the same joint distribution w.r.t. φ,ψ,θ and form φ,ψ,θ-indentedly independent sets, Lemma 3.12 implies for all fixed $j\in\mathbb{N}$ and all $i_1,\dots,i_j\in I$, that the moment $\varphi(X_{r_1}^{i_1}\dots X_{r_j}^{i_j})$ depends only on the ordered partition $\pi\in OP(j)$ corresponding to $(r_1,\dots,r_j)\in\mathbb{N}^j$. We may denote $\varphi(X_{r_1}^{i_1}\dots X_{r_j}^{i_j})=:\varphi(\pi;i_1,\dots,i_j)$. The reasoning follows now the argument from [15-17], in view of the other new lemmata from the previous section. We display it (as a sort of leitmotif) only for the reader's convenience.

Thus,

$$\varphi(S_N^{i_1}\dots S_N^{i_j})=\left(\tfrac{1}{\sqrt{N}}\right)^j\sum_{r_1,\dots,r_j=1}^{N}\varphi(X_{r_1}^{i_1}\dots X_{r_j}^{i_j})=\left(\tfrac{1}{\sqrt{N}}\right)^j\sum_{\pi\in OP(j)}C_N^{|\pi|}\varphi(\pi;i_1,\dots,i_j)\,,$$

as in [3,15-17, 20]; where $|\pi|$ denotes the number of blocks in π; and the number of representatives of the equivalence class corresponding to the involved partition $C_N^{|\pi|}:=N!/|\pi|!(N-|\pi|)!$ grows asymptotically like $N^{|\pi|}$ for large N. Lemma 3.4 implies that every partition with singletons has null contribution in the sum above. But the partitions without singletons have $|\pi| \le \frac{j}{2}$ blocks, and the limit of the factor $(\frac{1}{\sqrt{N}})^j C_N^{|\pi|}$ is 0, if $|\pi| < \frac{j}{2}$; and is $\frac{1}{|\pi|!}$, if $|\pi| = \frac{j}{2}$. So $\lim_{N\to\infty} \varphi(S_N^{i_1} \ldots S_N^{i_j}) = \sum_{\pi \in OP_2(j)} \frac{1}{|\pi|!} \varphi(\pi; i_1, \ldots, i_j)$, because π is a pairing, if $\pi \in OP(j)$ has no singletons and its number of blocks is equal to $\frac{j}{2}$. Thus, the odd moments vanish, since $\pi \in OP_2(j)$ is empty, when j is odd. We may conclude, by Lemmata 3.7 and 3.10, because the crossing ordered pairings have null contribution in the previous sum, and, respectively, the non-crossing ordered pairings give the claimed contribution.□

Remarks 4.4 1) *If, in particular, the φ, ψ, θ-indentedly independent sets of (selfadjoint) random variables are additionally ordered-free independent w.r.t. (ψ, θ), we get the multivariate CLT for the indented independent w.r.t. (φ, ψ, θ), in Hasebe's sense, identically distributed quantum random variables.*

2) *If $\varphi = \psi$, we obtain the multivariate CLT for the ψ, θ- ordered independent identically distributed quantum random variables; and if the ψ, ψ, θ-indentedly independent sets of (selfadjoint) random variables are additionally θ, ψ, θ-indentedly independent, we get the multivariate CLT for the ordered-free independent w.r.t. (ψ, θ), in Hasebe's sense, identically distributed quantum random variables.*

3) *If $\theta = \psi$, we obtain the multivariate CLT for the φ, ψ- free independent identically distributed quantum random variables in* [3][15, Th. 4.2]*; and if the φ, ψ- free independent sets of (selfadjoint) random variables are additionally ψ-free independent, we get the multivariate CLT for the c-free independent w.r.t. (φ, ψ), in Bozejko-Leinert-Speicher's sense* [2], *identically distributed quantum random variables*. If $\varphi = \psi = \theta$, *we get the multivariate CLT for the free independent identically distributed quantum random variables* [28].

4) *Suppose, without loss of generality, the (*-)algebra involved in the quantum (*-)probability space $(A, \varphi, \psi, \theta)$ decomposes as a direct sum of vector spaces $A = \mathbb{C} \cdot 1 \oplus D$ s.t. D is another (*-) algebra including all (selfadjoint) random variables $X_r^i, i \in I$, $r \in \mathbb{N}$.*

i) *Take the (positive) linear functionals $\varphi | D$ and $\psi | D$, given by restriction to D, denote them φ and ψ, too, and consider that $\theta | D = 0$. Then we recover the multivariate CLT for φ, ψ- monotone independent identically distributed quantum random variables in the quantum (*-)probability space (D, φ, ψ) from* [17, Th 4.2].

If these φ, ψ- monotone independent sets of (selfadjoint) random variables are additionally ψ-monotone independent, we obtain the multivariate CLT for c-monotone (in Hasebe's sense)[6] *identically distributed quantum random variables; in particular, if $\varphi = \psi$, we get the multivariate CLT for monotone quantum random variables in* [16, Th 4.2].

ii) *Take the (positive) linear functionals $\varphi | D$, denoted by φ, and $\theta | D =: \phi$, given by restriction to D, and consider that $\psi | D = 0$. Then we recover the multivariate CLT for φ, ϕ- anti-monotone independent identically distributed quantum random variables in the quantum (*-)probability space (D, φ, ϕ) (see, e.g.,* [17]).

If these φ, ϕ- anti-monotone independent sets of (selfadjoint) random variables are additionally ϕ-anti-monotone independent, we obtain the multivariate CLT for c-anti-monotone (in Hasebe's sense)[6] *identically distributed quantum random variables; in particular, if $\varphi = \phi$, we get the multivariate CLT for anti-monotone quantum random variables in* [9][16, Th 4.2].

iii) *Take the (positive) linear functional* $\varphi\big|D =: \phi$ *given by restriction to* D, *and consider that both* $\psi\big|D = 0$ *and* $\theta\big|D = 0$. *Then we recover the multivariate CLT for Boolean quantum random variables in the quantum (*-)probability space* (D,ϕ) (*see, e.g.,* [15,17]).

5) *The hypothesis of being identically distributed for the involved random vectors may be replaced by the pair (i)&(ii) below, as in the classical, Boolean, (c-)(anti-)monotone* [11,16,17] *or (c-)free cases* [11,15, 28] (*see also* [5, 33], *for some short proofs*), *with essentially the same proof as above, but we do not detail this here:*

i) $\sup_{r\in\mathbb{N}}\left|\varphi(X_r^{i_1}...X_r^{i_j})\right| < \infty$, $\sup_{r\in\mathbb{N}}\left|\psi(X_r^{i_1}...X_r^{i_j})\right| < \infty$, *and* $\sup_{r\in\mathbb{N}}\left|\theta(X_r^{i_1}...X_r^{i_j})\right| < \infty$ (*for all* j, *and all* $i_1,...,i_j \in I$);

ii) *there exist* $q_{ij} = \lim_{N\to\infty}\frac{1}{N}\sum_{r=1}^{N}\varphi(X_r^i X_r^j)$ $r_{ij} = \lim_{N\to\infty}\frac{1}{N}\sum_{r=1}^{N}\psi(X_r^i X_r^j)$, *and* $s_{ij} = \lim_{N\to\infty}\frac{1}{N}\sum_{r=1}^{N}\theta(X_r^i X_r^j)$.

6) *The combinatorial description of the joint moments of a Gaussian family (: multivariate normal distribution) in terms of all pairings instead of all non-crossing pairings (as a semicircular family* [28] *in the free probability theory), or all interval pairings (as a Bernoulli family in the Boolean probability theory), or all (anti-)monotone pairings (as an arcsine family* [9,16] *in the (anti-)monotone probability theory) is often named the Isserlis formula* [18] *in the classical probability or mathematical statistics theory and the Wick formula in the quantum field theory* (*see, e.g.,* [30]). *By analogy, the above formula describing the joint moments of such an indented (in particular, ordered) Gaussian family may be interpreted as an indented (in particular, ordered) Isserlis-Wick formula. This generalizes all previous purely non-commutative Isserlis-Wick type formulae.* □

In the same way, we can obtain operator-valued versions of these facts or other generalizations, but we do expose these elsewhere.

ACKNOWLEDGEMENTS

I am deeply grateful to Acad. Ioan Cuculescu and to Acad. Marius Iosifescu for many valuable discussions on these and related topics and their essential support, which has made my work possible. I am deeply grateful to Acad. Gabriela Marinoschi for many valuable comments and suggestions on all these subjects and moral support. I am deeply grateful to Professor Florin Boca for his interest concerning my work, many stimulating ideas, and moral support. I am deeply grateful to Professor Ioan Stancu-Minasian for his opportune support and much beneficial advice. I am deeply indebted to Professors Cecil P. Grunfeld, Vasile Preda, Marius Radulescu and Gheorghita Zbaganu for many stimulating conversations and moral support.

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Received December 31, 2024