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A MULTIVARIATE CENTRAL LIMIT THEOREM FOR C-MONOTONE QUANTUM RANDOM VARIABLES

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Abstract. We prove a multivariate CLT in T.Hasebe's c-monotone probability theory [7,8], by generalizing our proof in [16] for the CLT in N. Muraki and Y.G. Lu's monotone probability theory [23,24,20,21], inspired by that in [15] for M. Bożejko and R. Speicher's CLT [4,3] in the c-free probability theory; which extended the combinatorial method described in [10,27] for the CLT in the frame of D.-V. Voiculescu's free probability theory [31-33].

Key words: monotone partition, quantum probability space, non-commutative distribution, φ , ψ -monotone independence, Isserlis-Wick type formula.

1. INTRODUCTION

D.-V. Voiculescu's prodigious free probability theory (see, e.g., [31-33], but also [10,27] for more information) marked a flourishing period in the quantum probability (: QP) domain and the related fields. See, e.g., [6,22,28] (but also [11]), as an introduction into this domain. The papers [30] and [25,26] by R. Speicher and, respectively, N. Muraki are among the important moments in this evolution. They concern the classification of the QP theories based on a quantum stochastic independence concept arising from a (quasi-) universal or natural product of quantum probability spaces depending or not on the order of the factors. Speicher demonstrated there are only three such theories, when that product is not order-dependent: R. L. Hudson and K. R. Parthasarathy's Boson or Fermion probability theory, the free probability theory, and Speicher and W. von Waldenfels' Boolean probability theory, corresponding to the tensor, free and Boolean product, respectively. Muraki proved there exist precisely five such theories if that product possibly depends on the order of its factors; the other two fundamental theories, additional to the aforementioned three, being Muraki [23,24] and Y.G.Lu's [20,21] monotone probability theory and its dual, the anti-monotone probability theory, based on the monotone and respectively, the anti-monotone independence, emerging from the monotone and, respectively, the anti-monotone product.

Important endeavours were undertaken in parallel to unify or generalize some of these fundamental theories.

M. Bożejko and Speicher [4] generalized the free product and independence with respect to two states, via a product of quantum probability spaces non-dependent on the order of its factors. Their c-free product and independence [4,3] generalize also the Boolean product and independence, respectively, unifying the free and Boolean probability theory.

By parallelizing Bożejko and Speicher's c-free probability theory, T. Hasebe introduced [7] a generalization of the (anti-)monotone independence with respect to two states named the c-(anti-)monotone independence. As expected, this arises from a product of quantum probability spaces, called the c-(anti-) monotone product, dependent on the order of its factors. Consequently, if a_1 and a_2 are (c-) monotone independent random variables, it does not imply that a_2 and a_1 are, too.

Hasebe's c-(anti-)monotone probability theory (see, e.g., [7, 8] and the references therein) is a dynamic research topic generalizing and unifying the (anti-)monotone and Boolean probability theory. Hasebe proved a univariate CLT in this frame for identically distributed random variables, with a Kesten (more generally, free Meixner) distribution (see, e.g., [11]) as limit. In analogy to Bożejko-Speicher theory again, the combinatorial structure of the c-(anti-)monotone independence is governed by the lattice of the (anti-) monotone partitions (which are ordered non-crossing ones), but it must distinguish between the outer and the inner blocks of such a partition. Notably, R. Lenczewski [18] determined the product of graphs (i.e., the c-comb graphs) corresponding to the c-monotone independence, by generalizing the comb graphs through which L. Accardi, A. Ben Ghorbal and N. Obata [1] performed an important connection between Muraki-Lu monotone probability theory and the famous theory of Bose-Einstein condensation via the monotone CLT. See, e.g., [11] for the correspondence between the tensor, free, Boolean and monotone independence and the Cartesian, free, star and comb product of graphs, respectively, and further information.

In the present Note, we prove the multivariate CLT for φ,ψ-monotone random variables in Hasebe's theory, by generalizing, with respect to an additional state, our elementary proof from [16] for the CLT in Muraki-Lu monotone probability theory. This was inspired by the proof from [15] of the CLT for φ,ψ-free random variables in Bożejko-Speicher theory; which extended the combinatorial moment method presented in [10] or [27] for the free CLT. The setting is essentially that from [16], but the simple random variables are slightly more complicated now, because the quantum probability space is endowed with a pair of states φ,ψ as in [4,3,15]. This time, we concentrate on the occurrence of internal peaks given by interval blocks in the ordered partition associated to a product of ψ-centered φ,ψ-monotone independent random variables; via the weak independence in the sense of [5,12] again. The alternative proof by cumulants is shorter. Other limit theorems can be proved. We will expose these elsewhere.

2. PRELIMINARIES

We repeat for the reader's convenience some well-known general information as in, e.g., [2,11,15,16, 19,25-27], instead of sending directly to these references. (We abbreviate 'such that' by 's.t.', and 'with respect to' by 'w.r.t'). Let S be a finite totally ordered set (w.r.t. $<$). Denote by $P(S)$ the partitions of S; call blocks the non-empty subsets defining a partition. If S is a disjoint union of non-void subsets S_i, and $\pi \in P(S)$ s. t. $\pi = \cup \pi_i$, with some $\pi_i \in P(S_i)$, we write $\pi = \coprod \pi_i$. If, for instance, $S = \{s_1,...,s_n\}$, with $[s_1 < ... < s_n]$, we say $\pi \in P(S)$ is irreducible, when π does not factorize as $\pi_1 \coprod \pi_2$, with $\pi_i \in P(S_i)$, where $S_1 = \{s_1,...,s_p\}$ and $S_2 = \{s_{p+1},...,s_n\}$ are disjoint sets.We call pairing a partition in which every block has exactly two elements. For $k,l \in S$, denote by $k \sim_\pi l$ the fact that k and l belong to the same block of $\pi \in P(S)$. Recall that a partition π is called crossing if there are $k_1 < l_1 < k_2 < l_2$ in S s.t. $k_1 \sim_\pi k_2 \nsim_\pi l_1 \sim_\pi l_2$; otherwise, π is non-crossing. When π is non-crossing, and V is a block of π, say V is inner, if there exist another block W of π, and $k,l \in W$, s. t. $k < v < l$, for all $v \in V$, denoting this by $W \prec V$; otherwise, say V is outer. Denote by $\mathrm{I}(\pi)$, and $\mathrm{O}(\pi)$ the inner, and, respectively, outer blocks of π. Recall that a non-crossing partition π is called an interval partition if $\mathrm{I}(\pi)$ is empty. Denote by $NC(S)$, $P_2(S)$, $NC_2(S)$ and $I_2(S)$ the non-crossing partitions, the pairings, the non-crossing pairings, and the interval pairings of S, respectively.

An ordered (coloured) partition of S is a partition $\pi = (P_1,...,P_r)$ of S endowed with an ordering (colouring) (: a permutation) of its blocks [19,25]; s being the order (colour) of the block P_s. If $\pi \in P(S)$, there exist $|\pi|!$ ways to order (colour) π, where $|\pi|$ is the number of blocks of π. We symbol the block as $P_\bullet$ when its order (colour) is not specified. Denote by $OP(S)$ the ordered (coloured) partitions of S. For any $1 \le q \le r$, we may consider any $\pi = (P_1,...,P_r) \in OP(S)$ as $\{P_1,...,P_q,P_{s_1},...,P_{s_j}\} \in P(S)$, with some $q+1 \le s_1,...,s_j \le r$, by neglecting the ordering (colouring) of its blocks after the block P_q. Denote by $ONC(S)$ the ordered (coloured) non-crossing partitions of S.

A monotone partition [19,25] of S is a partition $\pi = (P_1,...,P_r) \in ONC(S)$ s.t. its ordering (colouring) is order-reflecting: for any pair of blocks $P_k \prec P_l$ in π, it holds $k<l$. If $\pi \in ONC(S)$ is not monotone, we say π is non-monotone. We denote by $M_2(S)$ the monotone pairings of S.

When S has m elements, abbreviate by $P_2(m)$, $NC_2(m)$, $I_2(m)$, $OP(m)$, $OP_2(m)$, $ONC_2(m)$, and $M_2(m)$, the pairings, non-crossing pairings, interval pairings, the ordered (coloured) partitions, pairings, non-crossing pairings, and the monotone pairings of S, respectively. $P_2(m)$ is empty if m is odd. Recall that each non-crossing partition of $\{1,...,m\}$ has at least an interval; i.e., a block of consecutive indices which may be a singleton (:block having a single element). Remind the cardinality of $P_2(2p)$ or $NC_2(2p)$ or $M_2(2p)$ equals the corresponding moment of a standard Gauss, respectively, semi-circular Wigner or (by a factor of $p!$) arcsine distribution; i.e., $(2p)!!$, respectively the Catalan number $c_p := (2p)!/p!(p+1)!$ or $(2p)!!$, too.

We consider a *- algebra as a (complex) associative algebra with an involution * (i.e. a conjugate linear anti-automorphism). A linear functional φ of a *- algebra A is positive if $\varphi(a^*a) \ge 0$, for all $a \in A$. Let A be a (complex) (*-) algebra, and φ, ψ be two states; i.e., linear (positive) functionals of A. We interpret (A,φ), (A,ψ) or (A,φ,ψ) as quantum (*-) probability spaces, and the elements of A as quantum random variables in view of [31,27]. Let I be an index set and $\mathbb{C} < \xi_i, i \in I >^\circ$ be the (*-) algebra (without a unit) freely generated by the complex field $\mathbb{C}$ and the non-commuting indeterminates $\xi_i, i \in I$. Let $a = (a_i)_{i \in I}$ be such a random vector with all (self-adjoint) $a_i \in A$. The non-commutative joint distribution of a w.r.t. φ is $\varphi_a := \varphi \circ \tau_a$, where $\tau_a : \mathbb{C} < \xi_i, i \in I >^\circ \to A$ is the unique (*-) homomorphism s.t. $\tau_a(\xi_i) = a_i$. The scalars $\varphi(a_{i_1}...a_{i_j})$ are viewed as the joint moments of a w.r.t. φ.

If $a_N = (a_N^i)_{i \in I}$ and $a = (a_i)_{i \in I}$ are random vectors in some quantum probability spaces (A_N, φ_N) and (A,φ), we say $(a_N)_N$ converges in distribution to a, denoting $a_N \xrightarrow{distr} a$, if for all $j \ge 1$, and all $i_1,...,i_j \in I$, $\lim_{N\to\infty} \varphi_N(a_N^{i_1}...a_N^{i_j}) = \varphi(a_{i_1}...a_{i_j})$. When $a \in A$ and $\varphi(a) = 0$, say a is centered w.r.t. φ, or φ-centered. When a is centered w.r.t. φ, ψ, say it is φ, ψ-centered.

If I is totally ordered, $i_1,...,i_n \in I$ and $\{i_1,...,i_n\} = \{k_1,...,k_r\}$ with $k_1 < ... < k_r$, the ordered (coloured) partition corresponding to $j \mapsto i_j$ is $(P_1,...,P_r) \in OP(n)$ given by $P_j = \{s; i_s = k_j\}$ [19]. When $A_i \subset A$, $i \in I$ are subalgebras, and $w = a_1 \cdots a_n \in A$ is a random variable, s.t. all $a_j \in A_{i_j}$, for $i_1,...,i_n \in I$, the ordered (coloured) partition associated to w is that corresponding to $j \mapsto i_j$.

If $i_1 > i_2$ or $i_{n-1} < i_n$, we say $j \mapsto i_j$ has i_1, respectively, i_n as marginal peaks , and a_1, respectively, a_n is marginal peak in w; when there exists $2 \le p < n$ with $i_{p-1} < i_p > i_{p+1}$ (respectively, $i_{p-1} > i_p < i_{p+1}$), we say $j \mapsto i_j$ has i_p as internal peak (respectively, bottom), and a_p is an internal peak (respectively, a bottom) in w.When $j \mapsto i_j$ has internal peaks, we say a_p , respectively, a_q is the left, respectively, right internal peak in w, if $p := \min\{2 \le s \le n-1; i_{s-1} < i_s > i_{s+1}\}$ and $q := \max\{2 \le s \le n-1; i_{s-1} < i_s > i_{s+1}\}$.

We say $w = a_1 \cdots a_n \in A$, with $a_k \in A_{i_k}$, as before, is a simple random variable in (A,φ,ψ) if w is reduced (i.e., $k \mapsto i_k$ has not intervals: $i_1 \neq i_2 \neq \ldots \neq i_n$), calling n the length of w, and w has a φ-centered marginal peak or a φ,ψ-centered internal peak.

We say $(A_i)_{i\in I}$ has the φ-factorization property by marginal peaks if, for all $n \geq 2$, all $i_1 \neq \ldots \neq i_n$, and all $a_k \in A_{i_k}$, it holds:

i) $\varphi(a_1 \cdots a_n) = \varphi(a_1)(a_2 \cdots a_n)$, when $i_1 > i_2$; and ii) $\varphi(a_1 \cdots a_n) = \varphi(a_1 \cdots a_{n-1})\varphi(a_n)$, when $i_{n-1} < i_n$.

Remark 2.1 *Let $w = a_1 \cdots a_n \in A$ be reduced, with $a_j \in A_{i_j}$. If $(A_i)_{i\in I}$ has the φ-factorization property by marginal peaks, and $j \mapsto i_j$ is strictly monotone on $\{1,\ldots,n\}$, then $\varphi(w) = \varphi(a_1)\cdots\varphi(a_n)$.*□

We say $(A_i)_{i\in I}$ has the φ,ψ-decomposition property by internal peaks if, for all $i_1 \neq \ldots \neq i_n$, and all $a_k \in A_{i_k}$, it holds:

$\varphi(a_1 \cdots a_n) = \psi(a_p)\varphi(a_1 \cdots a_{p-1}a_{p+1} \cdots a_n) + [\varphi(a_p) - \psi(a_p)]\varphi(a_1 \ldots a_{p-1})\varphi(a_{p+1} \ldots a_n)$, whenever $2 \leq p < n$ and $i_{p-1} < i_p > i_{p+1}$. In this case, we call $a_1 \cdots a_{p-1}a_{p+1} \cdots a_n$ the subword obtained from w by excluding a_p via ψ.

The next definition concerning the notion of φ,ψ-monotone independence is inspired from [4,7,8,13-16,18]. The dual concept of φ,ψ-anti-monotone independence is defined by reversing the order on I.

Definition 2.2 *Let (A,φ,ψ) be a quantum probability space as above, and $A_i \subset A, i \in I$ be subalgebras. The family $(A_i)_{i\in I}$ is φ,ψ-monotone independent (or φ,ψ-monotone, for short), if it has the φ-factorization property by marginal peaks, and the φ,ψ-decomposition property by internal peaks. If $A \supset S_i$, $i \in I$ are subsets, then $(S_i)_{i\in I}$ is φ,ψ-monotone independent, if $(A_i)_{i\in I}$ is φ,ψ-monotone independent, A_i being the subalgebra of A generated by S_i.*□

In particular, the ψ,ψ-monotone independence is Muraki-Lu's monotone independence w.r.t. ψ [1, 7-9,11,16,19-21,23-26]. The conditionally monotone (or c-monotone, for brevity) independence w.r.t. (φ,ψ), considered in [7,8,18], involves both the monotone independence w.r.t. ψ, and the φ,ψ-monotone independence.

3. JOINT MOMENTS OF φ,ψ-MONOTONE QUANTUM RANDOM VARIABLES

Let in this section I be a totally ordered set, (A,φ,ψ) be a quantum probability space as before, and $A_i \subset A$, $i \in I$ be a family of φ,ψ-monotone independent subalgebras of A.

The assertion of 1) in the first lemma is immediate by Remark 2.1. If $i_1 < i_2$ (respectively, $i_{n-1} > i_n$), and a_p is the left (respectively, right) internal peak in w, then $j \mapsto i_j$ has no internal peaks on $\{1,\ldots,p-1\}$ (respectively, $\{p+1,\ldots,n\}$), and

$\varphi(a_1 \cdots a_{p-1}) = \varphi(a_1)\cdots\varphi(a_{p-1})$ (respectively, $\varphi(a_{p+1} \cdots a_n) = \varphi(a_{p+1})\cdots\varphi(a_n)$), by 1). Thus, 2) below follows via the φ,ψ-decomposition property by a_p. This easy lemma simplifies the argument for Lemmata 3.7-3.8 below.

Lemma 3.1 *Let $w = a_1 \cdots a_n \in A$ be reduced, s.t. every $a_j \in A_{i_j}$.*

1) *Suppose the map* $j \mapsto i_j$ *has no internal peaks. If* $i_1 < i_2$ *(respectively,* $i_{n-1} > i_n$*), then* a_n *(respectively,* a_1*) is a marginal peak in* w*, and* $\varphi(w) = \varphi(a_1)\cdots\varphi(a_n)$.

2) *Suppose the map* $j \mapsto i_j$ *has internal peaks.*

If $i_1 < i_2$, $\varphi(a_1) = 0$ *(respectively,* $i_{n-1} > i_n$, $\varphi(a_n) = 0$*),and* a_p *is the left (respectively, right) internal peak in* w*, then* $\varphi(w) = \psi(a_p)\varphi(a_1 \cdots a_{p-1}a_{p+1} \cdots a_n)$.

3) *If* w *is a simple random variable in* (A,φ,ψ)*, then* $\varphi(w) = 0$.□

We observe $(A_i)_{i\in I}$ is weakly independent in (A,φ) in the sense of [5,12]; remind the weak-independence has the meaning below.

Definition 3.2 *Let* (B,ω) *be a quantum probability space as before and* $B_i \subset B, i \in I$ *be subalgebras. The family* $(B_i)_{i\in I}$ *is weakly independent in* (B,ω)*, if* $\omega(x_1...x_n) = \omega(x_1...x_p)\omega(x_{p+1}...x_n)$*, for all* $n > p \ge 1$*, all* $i_j \in I$*, all* $x_j \in B_{i_j}$, s.t. *the sets* $\{i_1,...,i_p\}$ *and* $\{i_{p+1},...,i_n\}$ *are disjoint. If* $B \supset S_i$, $i \in I$ *are subsets, then* $(S_i)_{i\in I}$ *is weakly independent, if* $(B_i)_{i\in I}$ *is weakly independent;* B_i *being the subalgebra of* B *generated by* S_i.□

The second lemma extends [16, Lemma 3.2].

Lemma 3.3 $(A_i)_{i\in I}$ *is weakly independent in* (A,φ)*; i.e.,*

$\varphi(a_1 \cdots a_n) = \varphi(a_1 \cdots a_p)\varphi(a_{p+1} \cdots a_n)$, *for all* $n > p \ge 1$, *all* $i_j \in I$, *all* $a_j \in A_{i_j}$, s.t. *the sets* $\{i_1,...,i_p\}$ *and* $\{i_{p+1},...,i_n\}$ *are disjoint.*

Proof. It suffices to suppose $i_1 \ne i_2 \ne ... \ne i_n$. The φ-factorization property by marginal peaks implies the assertion for $n = 2$; and also for $n = 3$, $p = 2$ and $i_1 > i_2 < i_3$. If $n = 3$, $p = 2$ and $j \mapsto i_j$ is strictly monotone on $\{1,...,3\}$, the assertion results by Remark 2.1 . When a_2 is a peak in $a_1a_2a_3$, then $\varphi(a_1a_3) = \varphi(a_1)\varphi(a_3)$ and $\varphi(a_1a_2) = \varphi(a_1)\varphi(a_2)$ according to our assertion for $n = 2$; thus, $\varphi(a_1a_2a_3) = \varphi(a_1)\varphi(a_2)\varphi(a_3) = \varphi(a_1a_2)\varphi(a_3)$ due to the φ,ψ-decomposition property by a_2. And for $n = 3$ the case $p = 1$ follows as the $p = 2$ case.

Let $n > 3$. Suppose the statement true for any $r < n$. We may conclude by induction due to the inferences below.

If $i_1 > i_2$, we get $\varphi(a_1 \cdots a_n) = \varphi(a_1 \cdots a_p)\varphi(a_{p+1} \cdots a_n)$ through the φ-factorization property by marginal peaks (only; for $p = 1$), and the induction hypothesis for $p \ge 2$.

When $i_1 < i_2$ and $j \mapsto i_j$ has no internal peaks, then this map is strictly increasing on $\{1,...,n\}$; thus, $\varphi(a_1 \cdots a_n) = \varphi(a_1 \cdots a_p)\varphi(a_{p+1} \cdots a_n)$ by Remark 2.1 or Lemma 3.1.

Otherwise, consider an internal peak a_k in $a_1 \cdots a_n$. If $p < k-1$, the inductive hypothesis implies $\varphi(a_1 \cdots a_{k-1}a_{k+1} \cdots a_n) = \varphi(a_1 \cdots a_p)\ \varphi(a_{p+1} \cdots a_{k-1}a_{k+1} \cdots a_n)$ and $\varphi(a_1 \cdots a_{k-1}) = \varphi(a_1 \cdots a_p)\varphi(a_{p+1} \cdots a_{k-1})$; thus, the φ,ψ-decomposition property by a_k imposes

$\psi(a_k)\varphi(a_{p+1} \cdots a_{k-1}a_{k+1} \cdots a_n) + [\varphi(a_k) - \psi(a_k)]\varphi(a_{p+1} \cdots a_{k-1})\ \varphi(a_{k+1}...a_n) = \varphi(a_{p+1} \cdots a_n)$ and then $\varphi(a_1 \cdots a_n) = \varphi(a_1 \cdots a_p)\varphi(a_{p+1} \cdots a_n)$.

If $p \in \{k-1, k\}$, the φ,ψ-decomposition property by a_k again entails

$\varphi(a_1 \cdots a_n) = \psi(a_k)\varphi(a_1 \cdots a_{k-1}a_{k+1} \cdots a_n) + [\varphi(a_k) - \psi(a_k)]\varphi(a_1...a_{k-1})\varphi(a_{k+1}...a_n) =$

$\varphi(a_1...a_{k-1})\varphi(a_k)\varphi(a_{k+1}...a_n) = \varphi(a_1\cdots a_p)\varphi(a_{p+1}\cdots a_n)$, via the induction hypothesis and the φ-factorization property by marginal peaks.

If $p \geq k+1$, the inductive hypothesis implies as above

$\varphi(a_1\cdots a_{k-1}a_{k+1}\cdots a_n) = \varphi(a_1\cdots a_{k-1}a_{k+1}\cdots a_p)\ \varphi(a_{p+1}\cdots a_n)$ and

$\varphi(a_{k+1}\cdots a_n) = \varphi(a_{k+1}\cdots a_p)\ \varphi(a_{p+1}\cdots a_n)$; therefore, $\varphi(a_1\cdots a_n) = \varphi(a_1\cdots a_p)\varphi(a_{p+1}\cdots a_n)$ by applying twice the φ,ψ-decomposition property by a_k, for $a_1\cdots a_p$ and $a_1\cdots a_n$.□

For $w = a_1\cdots a_n \in A$ s.t. every $a_j \in A_{i_j}$, we say w has a_k as singleton when $i_j \neq i_k$, for any $j \neq k$.

The following statement slightly generalizes [16, Lemma 3.3].

Lemma 3.4 *Let $w = a_1\cdots a_n \in A$, s.t. every $a_j \in A_{i_j}$, and w has a singleton a_k which is φ,ψ-centered. Then $\varphi(w) = 0$.*

Proof. It suffices to suppose w is reduced. If $k \in \{1,n\}$, the assertion follows by the weak-independence (i.e., Lemma 3.3), and the φ-centeredness of a_k. It remains to consider $2 \leq k \leq n-1$. For $n = 3$, the assertion results via the weak-independence (i.e., Lemma 3.3), Lemma 3.1, and the φ,ψ-centeredness of a_k.

Suppose the statement true for any $a_1\cdots a_r \in A$ of length $r < n$; check it for $w = a_1\cdots a_n \in A$, as below.

If $i_{n-1} < i_n$, the φ-factorization property by a_n implies $\varphi(w) = \varphi(a_1\cdots a_{n-1})\varphi(a_n)$=0, because $\varphi(a_1\cdots a_{n-1}) = 0$ by the induction hypothesis or the weak-independence and the φ-centeredness of a_k.

If $i_{n-1} > i_n$, and $j \mapsto i_j$ has no internal peaks, then this map is strictly decreasing on $\{1,...,n\}$; so, $\varphi(a_1\cdots a_k\cdots a_n) = \varphi(a_1)\cdots\varphi(a_k)\cdots\varphi(a_n) = 0$ by Remark 2.1 or Lemma 3.1 and the φ-centeredness of a_k.

Alternatively, when the singleton a_k is even a peak in w, then w is a simple random variable in (A,φ,ψ), and $\varphi(w) = 0$, via Lemma3.1. Otherwise, consider the right internal peak a_p in w. Thus, $\varphi(a_1\cdots a_{p-1}a_{p+1}\cdots a_n)$ =0, by the inductive hypothesis. Moreover, the map $j \mapsto i_j$ is strictly decreasing on $\{p, p+1,...,n\}$ and $\varphi(a_{p+1}\cdots a_n) = \varphi(a_{p+1})\cdots\varphi(a_n)$ by the Remark 2.1. If $p-1 \geq k$, we get $\varphi(a_1\cdots a_{p-1}) = 0$ by the induction hypothesis or the weak-independence (i.e., Lemma 3.3); if $p+1 \leq k$, we get $\varphi(a_{p+1}\cdots a_n) = 0$ by the φ-centeredness of a_k. Consequently, $\varphi(w) = 0$ via the φ,ψ-decomposition property by a_p.□

If (A,φ,ψ) is a quantum probability space as before, and $x_1, x_2 \in A$ are random variables s.t. one of them is φ,ψ-centered, then $\varphi(x_1x_2) = k_2^{\varphi}(x_1,x_2)$, and $\psi(x_1x_2) = k_2^{\psi}(x_1,x_2)$; whenever, e.g., k_2^{φ} and k_2^{ψ} are the tensor/free/Boolean/monotone joint cumulants (see, e.g., [2, 27]) w.r.t. φ, ψ, respectively, of order two. In the sequel, we may use any of these choices.

In general, the scalars involved below $\overline{k}_{\pi}(x_1,...,x_n)$, for $\pi \in M_2(n)$, can be described as follows; parallelizing the c-free case (with $NC_2(n)$), see, e.g. [4, 15] .

1) If π has a single block, then that is an outer block of π, and $\overline{k}_{\pi}(x_1,x_2) := k_2^{\varphi}(x_1,x_2)$;

2) If $\pi = \sigma \coprod \rho$, with $\sigma \in M_2(i)$ and $\rho \in M_2(\{i+1,...,n\})$, then

$\overline{k}_{\pi}(x_1,...,x_n) := \overline{k}_{\sigma}(x_1,...,x_i)\cdot\overline{k}_{\rho}(x_{i+1},...,x_n)$;

3) If π contains the block $(1,n)$. , and the subpartition $\sigma = \pi \cap \{2,...,n-1\}$, then

$\overline{k}_\pi(x_1,...,x_n) := k_2^\varphi(x_1,x_n)k_\sigma(x_2,...,x_{n-1})$; where, more generally, for a subpartition σ of $\pi \in M_2(n)$, with $\sigma \in M_2(S)$, and $S = \{1,...,s\}$, the scalars $k_\sigma(x_i, i \in S)$, can be described in the following way.

1) If σ has a single block, then that is an inner block of π, and $k_\sigma(x_1,x_2) := k_2^\psi(x_1,x_2)$;

2) If $\sigma = \rho \coprod \tau$, with $\rho \in M_2(S_1)$, and $\tau \in M_2(S_2)$, then

$k_\sigma(x_1,...,x_s) := k_\rho(x_i, i \in S_1) \cdot k_\tau(x_i, i \in S_2)$.□

We illustrate the next lemma by the following crossing partitions in $OP_2(n)$ associated to $w = a_1 \cdots a_n \in A$.

Examples 3.5 1) For $n = 10$, let $\pi = ((1,6)_1,(2,3)_2,(5,10)_3,(8,9)_4,(4,7)_5)$. Then, in reduced form, $w = a_1c_2a_4a_5a_6a_7c_9a_{10}$,with $a_2a_3 =: c_2 \in A_{i_2}$ and $a_8a_9 =: c_9 \in A_{i_9}$ arising from the intervals $I_2 := (2,3)_2$, and $I_9 := (8,9)_4$ of π (but c_2, c_9 are not peaks), and a_4, a_7 as internal peaks. So, w is a simple random variable in (A,φ,ψ) , and $\varphi(w) = 0$, by , say, Lemma 3.1.

2) For $n = 16$, let $\sigma = ((1,10)_1,(9,16)_2,(6,13)_3,(2,3)_4,(7,8)_5,(14,15)_6,(4,5)_7,(11,12)_8)$. Now, $w = a_1c_2c_5a_6c_7a_9a_{10}c_{11}a_{13}c_{15}a_{16}$, under the reduced form, with $a_4a_5 =: c_5 \in A_{i_5}$, $a_7a_8 =: c_7 \in A_{i_7}$, $a_{11}a_{12} =: c_{11} \in A_{i_{11}}$, and $a_{14}a_{15} =: c_{15} \in A_{i_{15}}$ as internal peaks arising from the intervals $I_5 := (4,5)_7$, $I_7 := (7,8)_5$, $I_{11} := (11,12)_8$ and $I_{15} := (14,15)_6$ of σ, respectively. One uses Lemma 3.1. If one begins by excluding the left internal peak via ψ, one gets $\varphi(w) = \psi(c_5)\varphi(w_1)$,where w_1 is the subword of w corresponding to $\sigma_1 := \sigma \setminus \{I_5\}$ and having $a_2a_3 =: c_2 \in A_{i_2}$ as the left internal peak arised from $I_2 := (2,3)_4$. So, $\varphi(w_1) = \psi(c_2)\varphi(w_2)$, where w_2 is the subword of w_1 corresponding to $\sigma_2 := \sigma_1 \setminus \{I_2\}$ which has c_7 as the left internal peak. Then $\varphi(w_2) = \psi(c_7)\varphi(w_3)$, where w_3 is the subword of w_2 corresponding to $\sigma_3 := \sigma_2 \setminus \{I_7\}$; but, w_3 has a_6 as internal peak . Hence w_3 is a simple random variable in (A,φ,ψ), and $\varphi(w_3) = 0$, by, say, Lemma 3.1 . Another way, by excluding, for instance, the internal peaks from the right one, via ψ, one gets $\varphi(w) = \psi(c_{15})\psi(c_{11})\varphi(w_4)$; where w_4 is the subword of w corresponding to $\sigma_4 := \sigma \setminus \{I_{15}, I_{11}\}$, for which a_{13} is a peak. Hence w_4 is a simple random variable in (A,φ,ψ), and $\varphi(w_4) = 0$, too . □

Due to Lemma 3.1, we simply extend below [16, Lemma 3.4] and give another short proof for it.

Lemma 3.6 *Let* $w = a_1 \cdots a_n \in A$*, s.t. all* $a_j \in A_{i_j}$ *are* φ,ψ *-centered, and the ordered partition* π *associated to* w *is a crossing pairing. Then* $\varphi(w) = 0$.

Proof. In view of Lemma 3.3 (the weak independence) and Lemma 3.1, it remains to consider: π is irreducible, $i_1 < i_2$, $i_{n-1} > i_n$, w has (under its reduced form) only internal peaks arising from some interval blocks of π; and, under its reduced form, any subword obtained from w by excluding such an internal peak via ψ has no other internal peaks arising from interval blocks of π which does not occur in w.

Let $c_0, c_1, \ldots, c_r$ be all these internal peaks in the reduced form of w, considered from the left to the right internal peak; let I_k be the interval block of π corresponding to c_k. Lemma 3.1 successively implies for every c_k (with $w_1, \ldots, w_r$ under the reduced form):

$\varphi(w) = \psi(c_0)\varphi(w_1) = \psi(c_0)\psi(c_1)\varphi(w_2) = ... = \psi(c_0)\cdots\psi(c_{r-1})\varphi(w_r) = \psi(c_0)\cdots\psi(c_r)\varphi(w_0)$; where

$w_1, \ldots, w_r$ (having $c_1, \ldots, c_r$, respectively, as the left internal peak) and w_0 are the obvious subwords of w corresponding to the ordered subpartitions $\pi_1, \ldots, \pi_r$ and π_0 of π, respectively, given by $\pi_1 := \pi \setminus \{I_0\}$, $\pi_j := \pi_{j-1} \setminus \{I_{j-1}\}$, for $j = 2, ..., r$, and $\pi_0 := \pi_r \setminus \{I_r\}$.

Thus, $\pi_1, \ldots, \pi_r$ and π_0 are crossing pairings; and π_0 has no intervals giving internal peaks in w_0. Hence w_0 is a simple random variable in (A, φ, ψ); and $\varphi(w_0) = 0$, by Lemma 3.1. □

Another proof. The above proof can be slightly modified as follows. In view of Lemma 3.3 (the weak independence) and Lemma 3.1, it remains to consider: π is irreducible, $i_1 < i_2$, $i_{n-1} > i_n$, and w has (under its reduced form) only internal peaks arising from some interval blocks of π.

For $n = 8$, there are involved only the following crossing pairings: $\{(1,5)_1, (4,8)_2, (2,3)_s, (6,7)_t\}$, $\{(4,8)_1, (1,5)_2, (2,3)_s, (6,7)_t\}$, $\{(4,8)_1, (1,7)_2, (2,3)_s, (5,6)_t\} \in OP_2(8)$, with $3 \le s, t \le 4$. For each of them, the interval block $(2,3)_s$ gives the left internal peak c in the reduced form of w. The subword w' obtained from w by excluding c via ψ has a_2 as internal peak in the first two cases, and a_1 as marginal peak in the rest. Thus, w' is a simple random variable in (A, φ, ψ), and $\varphi(w) = \psi(c)\varphi(w') = 0$ by Lemma 3.1, always.

Let $n > 8$, and the statement true for all $p < n$. Then, for $w = a_1 \cdots a_n \in A$, the inferences below help to conclude by induction.

Let $a_r a_{r+1} =: c_r \in A_{i_r}$ be, for instance, the right internal peak in the reduced form of w, arising (as a singleton) in w from an interval $(r, r+1)_\bullet \in \pi$, with $i_{r-1} < i_r = i_{r+1} > i_{r+2}$. Then we may express $w = a_1 x c_r y a_n$; where x, y are void or arbitrary products of $a_j \in A_{i_j}$ with $\varphi(a_j) = 0 = \psi(a_j)$; but, y (as non-void factor in w) has no internal peaks. By the reducing of x, y, and Lemma 3.1, we get $\varphi(w) = \psi(c_r)\varphi(a_1 x y a_n)$. The ordered subpartition of π associated to $a_1 x y a_n$ is crossing and belongs to $OP_2(n-2)$. Hence $\varphi(w) = 0$, by the inductive hypothesis. □

We illustrate the next lemma by the following partitions in $ONC_2(n)$ associated to $w = a_1 \cdots a_n \in A$. (Compare with [16, Ex. 3.5].)

Examples 3.7 1) If $n = 4$, let $\pi_1 = ((2,3)_1, (1,4)_2)$ and $\pi_2 = ((1,4)_1, (2,3)_2)$, which are non-monotone and, respectively, monotone. Their interval gives $a_2 a_3 =: c_2 \in A_{i_2}$. Thus, $w = a_1 c_2 a_4$ as reduced word. For π_1, we get $\varphi(w) = \varphi(a_1 c_2)\varphi(a_4) = 0$, because a_4 is a marginal peak in w, and this is a simple random variable in (A, φ, ψ). For π_2, remark c_2 is an internal peak in w; so, Lemma 3.1 implies

$$\varphi(w) = \psi(c_2)\varphi(a_1 a_4) = k_2^{\psi}(a_2, a_3) k_2^{\varphi}(a_1, a_4) = \overline{k}_{\pi_2}(a_1, ..., a_4).$$

2) For $n = 6$, let π be any of the monotone pairings

$((1,6)_1, (2,3)_2, (4,5)_3)$, $((1,6)_1, (4,5)_2, (2,3)_3)$, and $((1,6)_1, (2,5)_2, (3,4)_3)$. For each of them, the interval block $(\cdot,\cdot)_3$ gives the unique internal peak c in the reduced form of w; thus, $\varphi(w) = \psi(c)\varphi(w')$, via Lemma 3.1, where w' is the subword obtained from w by excluding c via ψ. The computation of $\varphi(w')$ reduces to the above example for $n = 4$, implying $\varphi(w) = \overline{k}_\pi(a_1, ..., a_6)$ in each case.

3) For $n = 8$, let π be any of the following partitions: $((4,5)_1, (1,8)_2, (2,3)_3, (6,7)_4) \in ONC_2(8)$ (non-monotone); $\{(1,8)_1, (2,5)_2, (3,4)_s, (6,7)_t\}$, $\{(1,8)_1, (4,7)_2, (2,3)_s, (5,6)_t\}$, $\{(1,8)_1, (2,7)_2, (3,4)_s, (5,6)_t\} \in NC_2(8)$, with $3 \le s, t \le 4$, and $((1,8)_1, (6,7)_2 (2,5)_3, (3,4)_4)$, $((1,8)_1, (2,3)_2, (4,7)_3, (5,6)_4) \in ONC_2(8)$ (monotone). For each of them, the interval block $(\cdot,\cdot)_4$ gives the left or right internal peak c in the reduced form of w; so, $\varphi(w) = \psi(c)\varphi(w'')$, via Lemma 3.1 again; where w'' is the subword obtained from w by excluding c via

ψ. Then w'' has a marginal peak in the non-monotone case, being a simple random variable in (A,φ,ψ), and $\varphi(w'')=0$ via Lemma 3.1. But, for the monotone cases, the computation of $\varphi(w'')$ reduces to the examples before for $n=6$; thus one finally gets $\varphi(w)=\bar{k}_\pi(a_1,...,a_8)$ always.□

Due to Lemma 3.1 again, we easily get the adequate extension of [16, Lemma 3.6] by the same argument.

Lemma 3.8 *Let $w=a_1\cdots a_n\in A$, s.t. all $a_j\in A_{i_j}$ are φ,ψ-centered, and the ordered partition π associated to w is a non-crossing pairing. Then $\varphi(w)=0$, if π is not monotone; but $\varphi(w)=\bar{k}_\pi(a_1,...,a_n)$, if π is monotone.*

Proof. In view of Lemma 3.3 (the weak independence), we may consider $(1,n)_\cdot\in\pi$. If, under its reduced form, w has marginal peaks, or $i_1<i_2$, $i_{n-1}>i_n$, and w has an internal peak a_p that does not arise from an interval block of π, then w is (under its reduced form) a simple random variable in (A,φ,ψ); and observe: π has a pair of blocks $(k,\cdot)_\cdot\prec(k+1,\cdot)_\cdot$, with $k\in\{1,p\}$, or $(l,\cdot)_\cdot\prec(l-1,\cdot)_\cdot$, with $l\in\{p,n\}$, for which the colouring does not reflect the order. Thus, π is not monotone then, and $\varphi(w)=0$, by Lemma 3.1.

Therefore, the assertion being trivial for $n=2$, it remains to consider that $(1,n)_\cdot\in\pi$, $i_1<i_2$, $i_{n-1}>i_n$, and w has (under its reduced form) only internal peaks arising from some interval blocks of π. Note that, for any pair of blocks $P_k\prec P_l$ in π involving such an interval block giving the left (right) internal peak in w, the colouring is order-reflecting: $k<l$.

For $n=4,6$, see Examples 3.7. For $n=8$, there are only the following pairings, besides of the partitions in Examples 3.7:
$((4,5)_1,(1,8)_2,(6,7)_3,(2,3)_4)\in ONC_2(8)$ (non-monotone); $\{(1,8)_1,(2,3)_p,(4,5)_q,(6,7)_r\}\in NC_2(8)$, with $2\le p,q,r\le 4$, and $((1,8)_1,(2,7)_2,(3,6)_3,(4,5)_4)\in ONC_2(8)$ (monotone). For each of these pairings, the interval block $(\cdot,\cdot)_4$ also gives the left or right internal peak c in the reduced form of w. Thus $\varphi(w)=\psi(c)\varphi(w')$ by Lemma 3.1.Then the subword w' obtained from w by excluding c via ψ has a marginal peak in the non-monotone case again; when w' is a simple random variable in (A,φ,ψ), and $\varphi(w')=0$ via Lemma 3.1. And for the monotone cases, the computation of $\varphi(w')$ reduces again to the three cases for $n=6$ from Examples 3.7; thus, one finally gets $\varphi(w)=\bar{k}_\pi(a_1,...,a_8)$ always.

Let $n>8$. Suppose the assertion true for all $p<n$. To conclude by induction, remark the next facts.

Let $a_la_{l+1}=:c_l\in A_{i_l}$ be, for instance, the left internal peak in (the reduced form of) w, arising (as a singleton) in w from an interval $(l,l+1)_\cdot\in\pi$, with $i_{l-1}<i_l=i_{l+1}>i_{l+2}$. Therefore we may express $w=a_1xc_lya_n$; where x,y are void or arbitrary products of $a_j\in A_{i_j}$ with $\varphi(a_j)=0=\psi(a_j)$; but, x (as non-void factor in w) has no internal peaks. After reducing x and y, we infer again $\varphi(w)=\psi(c_l)\varphi(a_1xya_n)$ by Lemma 3.1.The ordered subpartition of π associated to a_1xya_n belongs now to $ONC_2(n-2)$.

If π is not monotone, this ordered subpairing of π is not monotone; hence $\varphi(w)=0$, by the inductive hypothesis. If π is monotone, this ordered subpairing of π is monotone, too. We may proceed as in [16, Lemma 3.6]. Let $\rho\in M_2(\{2,...,n-1\}\setminus\{l,l+1\})$ be the ordered sub-partition of π associated to xy. Since $\{(1,n)_\cdot\}\coprod\rho=:\sigma\in M_2(\{1,...,n\}\setminus\{l,l+1\})$ is the ordered sub-partition of π associated to a_1xya_n, the induction hypothesis implies

$\varphi(a_1 xya_n) = \overline{k}_\sigma(a_1, a_2, ..., a_{l-1}, a_{l+2}, ..., a_{n-1}, a_n) = k_2^\varphi(a_1, a_n) k_\rho(a_2, ..., a_{l-1}, a_{l+2}, ..., a_{n-1})$.

Thus,

$\varphi(w) = k_2^\psi(a_l, a_{l+1})\, k_2^\varphi(a_1, a_n) k_\rho(a_2, ..., a_{l-1}, a_{l+2}, ..., a_{n-1}) = \overline{k}_\pi(a_1, ..., a_n)$; because $\pi = \{(1, n).\} \coprod \tau$, with $\tau := \rho \coprod \{(l, l+1).\}$ and $\tau \in M_2(\{2, ..., n-1\})$.□

Lemma 3.9 *Let* $a_i = (a_i^s)_{s\in S}, i \in I$ *be random vectors in a probability space* (A, φ, ψ), *such that* $\{a_i^s, s \in S\} \subset A, i \in I$ *are* φ, ψ*-monotone independent sets of random variables in* (A, φ, ψ), *and* $a_i = (a_i^s)_{s\in S}, i \in I$ *have the same joint distribution w.r.t.* φ, ψ. *Then the joint moments of* $(a_i)_{i\in I}$ *w.r.t.* φ *are invariant under order-preserving injective maps; i.e., for all* n, *all* $s_1, ..., s_n \in S$, *all* $i_1, ..., i_n \in I$ *and all order-preserving injection* $\sigma: \{i_1, ..., i_n\} \to I$, *it holds* $\varphi(a_{i_1}^{s_1} ... a_{i_n}^{s_n}) = \varphi(a_{\sigma(i_1)}^{s_1} ... a_{\sigma(i_n)}^{s_n})$.

Proof. Since $a_i, i \in I$ are identically distributed w.r.t. φ, we get the statement if all i_k are equal. Otherwise, assume the statement true for any $r < n$.

Let consider $k \mapsto i_k$ has not intervals.

If $i_{n-1} < i_n$, the statement for n results from the φ-factorization property by the marginal peaks i_n and $\sigma(i_n)$, the inductive hypothesis for $n-1$, and the hypothesis cited above.

If $i_{n-1} > i_n$, and $k \mapsto i_k$ has no internal peaks, then this map and $k \mapsto \sigma(i_k)$ are strictly decreasing; so, the statement for n issues from Remark 2.1 or Lemma 3.1 and the same non-inductive hypothesis cited above.

If $i_{n-1} > i_n$, but $k \mapsto i_k$ has an internal peak $2 \le p < n$ s.t. $i_{p-1} < i_p > i_{p+1}$, then the statement for n follows via the φ, ψ-decomposition property by the internal peaks i_p and $\sigma(i_p)$, and the inductive hypothesis for $n-1$, $p-1$ and $n-p$; because $a_i, i \in I$ are identically distributed *w.r.t.* ψ, φ.

When $k \mapsto i_k$ has intervals, the statement for n results by the same argument as before, after a reducing of the random variables from both sides. Therefore, the statement being clear for $n \le 3$, we conclude by induction.□

4. C-MONOTONE GAUSSIAN FAMILY AND MULTIVARIATE CLT

Let I be an arbitrary index set. We remind a scalar matrix $q = \{q_{ij}\}_{i,j\in I}$ is positive if and only if $\sum_{k,l=1}^{n} q_{i_k, i_l} \overline{\lambda}_k \lambda_l \ge 0$, for all n, all $i_1, ..., i_n \in I$, and all $\lambda_1, ..., \lambda_n \in \mathbb{C}$.

The following definition is inspired from [2, 4, 5, 7, 12-16, 27]. If the two scalar matrices are the same, we recover the notion of monotone Gaussian family (see, e.g. [16, Def 4.1]). In particular, when r is 0 , we get (from [15, Def 4.1], too) the notion of Bernoulli (: Boolean Gaussian) family of covariance q, involving the interval pairings $I_2(j)$; an empty product being equal to 1 by convention.

Definition 4.1 *Let* $q = \{q_{ij}\}_{i,j\in I}$ *and* $r = \{r_{ij}\}_{i,j\in I}$ *be (positive) scalar matrices. Let* (A, φ) *be a quantum (*-) probability space. A family of (selfadjoint) random variables* $g = (g_i)_{i\in I}$ *in this is called a centered c-monotone Gaussian family of covariances* q *and* r, *if its distribution is of the following form, for all* $j \in \mathbb{N}$ *and all* $i_1, ..., i_j \in I$ *:*

$$\varphi(g_{i_1}...g_{i_j}) = \sum_{\pi\in M_2(j)} \tfrac{1}{|\pi|!}\overline{k}_\pi(g_{i_1},...,g_{i_j})\text{; where } \overline{k}_\pi(g_{i_1},...,g_{i_j}) := \prod_{(k,l)_\bullet\in \mathrm{O}(\pi)} q_{i_k i_l} \prod_{(k,l)_\bullet\in \mathrm{I}(\pi)} r_{i_k i_l}\,. \square$$

Theorem 4.2 *Let* (A,φ,ψ) *be a quantum (*-)probability space, and* $\{X_r^i, i\in I\}\subset A$, $r\in\mathbb{N}$ *be a sequence of* φ,ψ*-monotone independent sets of (selfadjoint) random variables in this, s.t.* $X_r = (X_r^i)_{i\in I}$ *has the same joint distribution for all* $r\in\mathbb{N}$*, and all variables are centered, both w.r.t.* φ,ψ*. Consider, for every* $N\geq 1$*, the sums* $S_N^i := \frac{1}{\sqrt{N}}\sum_{r=1}^{N} X_r^i \in A$*, and* $S_N := (S_N^i)_{i\in I}$ *as random vector in* (A,φ)*. Denote the covariances of the variables w.r.t.* φ,ψ *by* $q=\{q_{ij}\}_{i,j\in I}$ *and* $r=\{r_{ij}\}_{i,j\in I}$*; i.e.,* $q_{ij} := \varphi(X_r^i X_r^j)$*, and* $r_{ij} := \psi(X_r^i X_r^j)$*. Then* $S_N \xrightarrow{distr} g$*; where* $g=(g_i)_{i\in I}$ *is a centered c-monotone Gaussian family of (positive) covariances* q *and* r*.*

Proof. Since all X_r have the same joint distribution w.r.t. φ,ψ and form φ,ψ-monotone independent sets, Lemma 3.9 implies for all fixed $j\in\mathbb{N}$ and all $i_1,...,i_j\in I$, that the moment $\varphi(X_{r_1}^{i_1}...X_{r_j}^{i_j})$ depends only on the ordered partition $\pi\in OP(j)$ corresponding to $(r_1,...,r_j)\in\mathbb{N}^j$. We may denote $\varphi(X_{r_1}^{i_1}...X_{r_j}^{i_j}) =: \varphi(\pi;i_1,...,i_j)$. The reasoning repeats now the argument from [15,16], in light of the other lemmata from the previous section. We expose it for the reader's convenience.

Thus,

$$\varphi(S_N^{i_1}...S_N^{i_j}) = (\tfrac{1}{\sqrt{N}})^j \sum_{r_1,...,r_j=1}^{N} \varphi(X_{r_1}^{i_1}...X_{r_j}^{i_j}) = (\tfrac{1}{\sqrt{N}})^j \sum_{\pi\in OP(j)} C_N^{|\pi|}\varphi(\pi;i_1,...,i_j)\,,$$

as in [4,15,16,19]; where $|\pi|$ denotes the number of blocks in π; and the number of representatives of the equivalence class corresponding to the involved partition $C_N^{|\pi|} := N!/|\pi|!(N-|\pi|)!$ grows asymptotically like $N^{|\pi|}$ for large N. Lemma 3.4 implies that every partition with singletons has null contribution in the sum above. But the partitions without singletons have $|\pi|\leq\frac{j}{2}$ blocks, and the limit of the factor $(\frac{1}{\sqrt{N}})^j C_N^{|\pi|}$ is 0, if $|\pi|<\frac{j}{2}$; and is $\frac{1}{|\pi|!}$, if $|\pi|=\frac{j}{2}$. So $\lim_{N\to\infty}\varphi(S_N^{i_1}...S_N^{i_j}) = \sum_{\pi\in OP_2(j)}\frac{1}{|\pi|!}\varphi(\pi;i_1,...,i_j)$, because π is a pairing, if $\pi\in OP(j)$ has no singletons and its number of blocks is equal to $\frac{j}{2}$. Thus, the odd moments vanish, since $\pi\in OP_2(j)$ is empty, when j is odd. We may conclude, by Lemmata 3.6 and 3.8, because the crossing ordered pairings or the non-monotone non-crossing ordered pairings have null contribution in the previous sum, and, respectively, the monotone pairings give the claimed contribution.$\square$

Remarks 4.3 1) If, in particular, the φ,ψ-monotone sets of (selfadjoint) random variables are additionally ψ-monotone, we get the multivariate CLT for c-monotone identically distributed quantum random variables.

2) If $\varphi=\psi$, we obtain the multivariate CLT for monotone quantum random variables in [16, Th 4.2].

3) If $\psi=0$, we get the multivariate CLT for Boolean quantum random variables (as we do from [15, Th 4.2]).

4) The hypothesis of being identically distributed for the involved random vectors may be replaced by the pair (i)&(ii) below, as in the classical, Boolean, monotone [11,15,16] or (c-)free cases [11,15,27] (see also [6,31], for simple proofs), with essentially the same proof as above, but we do not detail this here:

i) $\sup_{r\in\mathbb{N}}\left|\varphi(X_r^{i_1}...X_r^{i_j})\right|<\infty$, $\sup_{r\in\mathbb{N}}\left|\psi(X_r^{i_1}...X_r^{i_j})\right|<\infty$ (for all j, and all $i_1,...,i_j\in I$);

ii) there exist $q_{ij} = \lim_{N\to\infty} \frac{1}{N}\sum_{r=1}^{N}\varphi(X_r^i X_r^j)$ and $r_{ij} = \lim_{N\to\infty} \frac{1}{N}\sum_{r=1}^{N}\psi(X_r^i X_r^j)$.

5) By reversing the order on I, we get the assertions corresponding to the multivariate CLT for the φ,ψ - anti-monotone or c-anti-monotone random variables (in terms of bottoms instead of peaks) via the anti-monotone partitions.

6) The combinatorial description of the joint moments of a Gaussian family (: multivariate normal distribution) in terms of all pairings instead of all non-crossing pairings (as a semicircular family [27] in the free probability theory), or all interval pairings (as a Bernoulli family in the Boolean probability theory), or all monotone pairings (as an arcsine family [9,16] in the monotone probability theory) is often named the Isserlis formula [17] in the classical probability or mathematical statistics theory and the Wick formula in the quantum field theory (see, e.g., [29]). By analogy, the above formula describing the joint moments of such a c-(anti-) monotone Gaussian family may be interpreted as a c-(anti-) monotone Isserlis-Wick formula. □

In the same way, we can obtain operator-valued versions of these facts or other generalizations, but we do expose these elsewere.

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