Institutul de Statistică Matematică şi Matematică Aplicată
”Gheorghe Mihoc – Caius Iacob” al Academiei Române
Calea 13 Septembrie nr. 13, sector 5, 050711 Bucureşti
Tel. 021 318 2433 Fax 021 318 2439
E-mail: office@ismma.ro

ISMMA Preprint Series

No. 1/2023

A new proof of the central limit theorem for monotone quantum random variables

Valentin Ionescu

“Gheorghe Mihoc – Caius Iacob” Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy

Recommended by

Gheorghiţă Zbăganu

A NEW PROOF OF THE CENTRAL LIMIT THEOREM FOR MONOTONE QUANTUM RANDOM VARIABLES

Valentin IONESCU

Gheorghe Mihoc-Caius Iacob Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Casa Academiei Române, Calea 13 Septembrie no. 13, 050711 Bucharest, Romania. E-mail: vionescu@csm.ro

Abstract. We prove a multivariate CLT in N. Muraki and Y.G.Lu 's monotone probability theory [18,19, 20,16,17] inspired by our proof in [13] for M. Bożejko and R. Speicher's CLT [3] in the c-free probability theory, which extended the combinatorial method exposed by F. Hiai and D. Petz [9] or A. Nica and R. Speicher [22] for CLT in the setting of D.-V. Voiculescu's free probability theory [26, 27].

Key words: monotone partition, quantum probability space, non-commutative distribution, monotone independence, Isserlis-Wick type formula.

1. INTRODUCTION

During the prodigious development of D.-V. Voiculescu's free probability theory [26, 27], R. Speicher proved [25] the existence of only three theories in the quantum probability domain (see, e.g., [5] as an introduction into), based on an analogue of the stochastic independence from the classical (:Kolmogorovian) probability theory, as the central concept, arising from a product of quantum probability spaces (: universal, in a certain natural sense, associative and not depending on the order of its factors); these being R. L. Hudson and K. R. Parthasarathy's Boson or Fermion probability theory, the free probability theory, and R. Speicher and W.von Waldenfels' Boolean probability theory.

But then, by previous investigations on some toy Fock spaces, concerning either a non-commutative de Moivre-Laplace theorem or CLTs for the quantum Bernoulli processes, N. Muraki (on his monotone Fock space) [18] and Y.G.Lu (on the interacting free Fock space)[16,17] independently discovered the monotone quantum Brownian motion: this being governed by a probability law which is neither Gauss, nor semi-circular Wigner, nor Bernoulli distribution, but a scaled symmetric arcsine distribution.

Muraki deduced [20] the quantum concept of stochastic independence (:the monotone independence) hidden in these arcsine de Moivre-Laplace theorem and Brownian motion, and proved a univariate CLT, in C*-algebraic frame, for monotone independent identically distributed random variables with a standard arcsine distribution as limit. He also constructed [19] the corresponding monotone product of quantum C*-probability spaces, and U.Franz proved [6] this product is universal in the same natural sense, associative, but depends on the order of the factors. Thus, if a_1 and a_2 are monotone independent random variables, it does not imply that a_2 and a_1 are, too.

Then, Muraki [21] revealed that the combinatorial structure of the monotone independence is governed by certain non-crossing ordered partitions (: the monotone partitions) and completed Speicher's result [25] by demonstrating there are exactly five quantum probability theories based on a corresponding notion of algebraic stochastic independence emerging from a product of quantum probability spaces which is universal, in a certain natural sense, associative and possibly depends on the order of its factors; by adding thus conclusively to the three fundamental theories before, the monotone probability theory and its dual (by reversing the order), the anti-monotone probability theory, which are as important as the other three.

In this sense, besides the papers already cited or the others [2, 7, 8, 10, 15, 24] and the bibliography therein, we mention only that L. Accardi, A. Ben Ghorbal and N. Obata [1] performed an important connexion between Muraki-Lu monotone probability theory and the famous theory of Bose-Einstein condensation from the condensed matter physics, by generalizing the comb lattices to comb graphs and via the monotone central limit theorem. For the present dynamic development of this fundamental quantum probability theory, see, e.g., the recent [7] and the rich list of references therein.

In this Note, we prove the multivariate CLT for monotone random variables in Muraki-Lu theory, inspired by our proof in [13] of the CLT in Bożejko-Speicher c-free probability theory, which extended the combinatorial moment method presented in [9] or [22] for the free CLT; thus, we focus on the occurence of the interval blocks in the ordered partition now associated to a product of centered monotone independent random variables, via the weak independence in the sense of [4,11]. The alternative proof by cumulants is shorter. Other limit theorems can be proved. We will detail these elsewhere.

2. PRELIMINARIES

We recall some well-known general information as in, e.g., [2,6,10,13,15,21,22]. (We abbreviate 'such that' by 's.t.', and 'with respect to' by 'w.r.t'). Let S be a finite totally ordered set (w.r.t. $<$). Denote by $P(S)$ the partitions of S; call blocks the non-empty subsets defining a partition. If S is a disjoint union of non-void subsets S_i, and $\pi \in P(S)$ s. t. $\pi = \cup \pi_i$, with some $\pi_i \in P(S_i)$, we write $\pi = \coprod \pi_i$. If, for instance, $S = \{s_1, ..., s_n\}$, with $s_1 < ... < s_n$, we say $\pi \in P(S)$ is irreducible, when π does not factorize as $\pi_1 \coprod \pi_2$, with $\pi_i \in P(S_i)$, where $S_1 = \{s_1, ..., s_p\}$ and $S_2 = \{s_{p+1}, ..., s_n\}$ are disjoint sets.We call pairing a partition in which every block has exactly two elements. For $k, l \in S$, denote by $k \sim_\pi l$ the fact that k and l belong to the same block of $\pi \in P(S)$. Recall that a partition π is called crossing if there are $k_1 < l_1 < k_2 < l_2$ in S s. t. $k_1 \sim_\pi k_2 \nsim_\pi l_1 \sim_\pi l_2$; otherwise, π is non-crossing. When π is non-crossing, and V is a block of π, say V is inner, if there exist another block W of π, and $k, l \in W$, s. t. $k < v < l$, for all $v \in V$, denoting this by $W \prec V$; otherwise, say V is outer.

An ordered (coloured) partition of S is a partition $\pi = (P_1, ..., P_r)$ of S endowed with an ordering (colouring) (: a permutation) of its blocks [15,21]; s being the order (colour) of the block P_s. If $\pi \in P(S)$, there exist $|\pi|!$ ways to order (colour) π, where $|\pi|$ is the number of blocks of π. We symbol the block as $P_\bullet$ when its order (colour) is not specified. Denote by $OP(S)$ the ordered (coloured) partitions of S. For any $1 \le q \le r$, we may consider any $\pi = (P_1, ..., P_r) \in OP(S)$ as $\{P_1, ..., P_q, P_{s_1}, ..., P_{s_j}\} \in P(S)$, with some $q + 1 \le s_1, ..., s_j \le r$, by neglecting the ordering (colouring) of its blocks after the block P_q. Denote by $ONC(S)$ the ordered (coloured) non-crossing partitions of S.

A monotone partition [15,21] of S is a partition $\pi = (P_1, ..., P_r) \in ONC(S)$ s.t. its ordering (colouring) is order-reflecting: for any pair of blocks $P_k \prec P_l$ in π, it holds $k < l$. Denote by $B(\pi)$ the blocks of π. If $\pi \in ONC(S)$ is not monotone, we say π is non-monotone. We denote by $M_2(S)$ the monotone pairings of S.

When S has m elements, abbreviate by $P_2(m)$, $NC_2(m)$, $OP(m)$, $OP_2(m)$, $ONC_2(m)$, and $M_2(m)$, the pairings, non-crossing pairings, the ordered (coloured) partitions, pairings, non-crossing pairings, and the monotone pairings of S, respectively. $P_2(m)$ is empty if m is odd. Recall that each non-crossing partition of $\{1,...,m\}$ has at least an interval; i.e., a block of consecutive indices which may be a singleton (:block having a single element).The cardinality of $P_2(2p)$ or $NC_2(2p)$ or $M_2(2p)$ equals the corresponding moment of a standard Gauss, respectively, semi-circular Wigner or (by a factor of $p!$) arcsine distribution; i.e., $(2p)!!$, respectively the Catalan number $c_p := (2p)!/p!(p+1)!$ or $(2p)!!$, too.

We consider a *- algebra as a (complex) associative algebra with an involution * (i.e. a conjugate linear anti-automorphism). A linear functional φ of a *- algebra A is positive if $\varphi(a^*a) \geq 0$, for all $a \in A$. Let A be a (complex) (*-) algebra, and φ be a linear (positive) functional of A. We interpret (A,φ) as a quantum (*-) probability space, and the elements of A as quantum random variables in view of [26, 27, 22]. Let I be an index set and $\mathbb{C} < \xi_i, i \in I >^\circ$ be the (*-) algebra (without a unit) freely generated by the complex field $\mathbb{C}$ and the non-commuting indeterminates $\xi_i, i \in I$. Let $a = (a_i)_{i\in I}$ be such a random vector with all (self-adjoint) $a_i \in A$. The non-commutative joint distribution of a w.r.t. φ is $\varphi_a := \varphi \circ \tau_a$, where $\tau_a : \mathbb{C} < \xi_i, i \in I >^\circ \to A$ is the unique (*-) homomorphism s.t. $\tau_a(\xi_i) = a_i$. The scalars $\varphi(a_{i_1}...a_{i_j})$ are viewed as the joint moments of a w.r.t. φ.

If $a_N = (a_N^i)_{i\in I}$ and $a = (a_i)_{i\in I}$ are random vectors in some quantum probability spaces (A_N, φ_N) and (A,φ), we say $(a_N)_N$ converges in distribution to a, denoting $a_N \xrightarrow{distr} a$, if for all $j \geq 1$, and all $i_1,...,i_j \in I$, $\lim_{N\to\infty} \varphi_N(a_N^{i_1}...a_N^{i_j}) = \varphi(a_{i_1}...a_{i_j})$. When $a \in A$ and $\varphi(a) = 0$, say a is centered w.r.t. φ, or centered, for short.

If I is totally ordered and $\{i_1,...,i_n\} = \{k_1,...,k_r\}$ with $k_1 < ... < k_r$, the ordered (coloured) partition corresponding to $j \mapsto i_j$ is $(P_1,...,P_r) \in OP(n)$ given by $P_j = \{s; i_s = k_j\}$ [15]. When $A_i \subset A$, $i \in I$ are subalgebras, and $w = a_1 \cdots a_n \in A$ is a random variable, s.t. all $a_j \in A_{i_j}$, for $i_1,...,i_n \in I$, the ordered (coloured) partition associated to w is that corresponding to $j \mapsto i_j$.

If $i_1 > i_2$ or $i_{n-1} < i_n$, we say $j \mapsto i_j$ has i_1, respectively, i_n as marginal peaks , and a_1, respectively, a_n is marginal peak in w; when there exists $2 \leq p < n$ with $i_{p-1} < i_p > i_{p+1}$ (respectively, $i_{p-1} > i_p < i_{p+1}$), we say $j \mapsto i_j$ has i_p as internal peak (respectively, bottom), and a_p is an internal peak (respectively, a bottom) in w.When $j \mapsto i_j$ has internal peaks, we say a_p , respectively, a_q is the left, respectively, right internal peak in w, if $p := \min\{2 \leq s \leq n-1; i_{s-1} < i_s > i_{s+1}\}$ and $q := \max\{2 \leq s \leq n-1; i_{s-1} < i_s > i_{s+1}\}$.

We say $w = a_1 \cdots a_n \in A$, with $a_k \in A_{i_k}$, as before, is a simple random variable in (A,φ), if w is reduced (i.e., $i_1 \neq i_2 \neq ... \neq i_n$) , calling n the length of w, and w has a centered (marginal or internal) peak.

Let (A,φ) be a quantum probability space as before, I be a totally ordered set, and $A_i \subset A$, $i \in I$ be a family of subalgebras of A.

We say the family $(A_i)_{i\in I}$ has the factorization property by marginal peaks in (A,φ), if, for all $n \geq 2$, all $i_1 \neq ... \neq i_n$, and all $a_k \in A_{i_k}$, it holds: i) $\varphi(a_1 \cdots a_n) = \varphi(a_1)\varphi(a_2 \cdots a_n)$, when $i_1 > i_2$; and ii) $\varphi(a_1 \cdots a_n) = \varphi(a_1 \cdots a_{n-1})\varphi(a_n)$, when $i_{n-1} < i_n$.

We say the family $(A_i)_{i\in I}$ has the factorization property by internal peaks in (A,φ), if, for all $i_1 \neq ... \neq i_n$, and all $a_k \in A_{i_k}$, it holds: $\varphi(a_1 \cdots a_n) = \varphi(a_p)\, \varphi(a_1 \cdots a_{p-1}a_{p+1} \cdots a_n)$, whenever $2 \leq p < n$ and $i_{p-1} < i_p > i_{p+1}$.

The next definition concerning the notion of monotone independence comes from [2,6,10,12,20, 21, 24]. The dual concept of anti-monotone independence is defined by reversing the order on I.

Definition 2.1 *Let* (A,φ) *be a quantum probability space as above, and* $A_i \subset A, i \in I$ *be subalgebras. The family* $(A_i)_{i\in I}$ *is monotone independent* (*or* monotone, *for short*) *in* (A,φ), *if it has the factorization property by (marginal, and internal) peaks. If* $A \supset S_i$, $i \in I$ *are subsets, then* $(S_i)_{i\in I}$ *is monotone independent in* (A,φ), *if* $(A_i)_{i\in I}$ *is monotone independent in* (A,φ), A_i *being the subalgebra of* A *generated by* S_i.□

Remark 2.2 *Let* $w = a_1 \cdots a_n \in A$ *be reduced, with all* $a_j \in A_{i_j}$.

1) *Assume* $(A_i)_{i\in I}$ *has the factorization property by marginal peaks in* (A,φ).

(i) *If* $j \mapsto i_j$ *is strictly monotone on* $\{1,...,n\}$, *then* $\varphi(w) = \varphi(a_1)\cdots\varphi(a_n)$;

(ii) *If* w *has a centered marginal peak, then* $\varphi(w) = 0$.

2) *If* $(A_i)_{i\in I}$ *is monotone independent and* w *is a simple random variable in* (A,φ), *then* $\varphi(w) = 0$.□

3. JOINT MOMENTS OF MONOTONE QUANTUM RANDOM VARIABLES

Let in this section (A,φ) be a quantum probability space as before, and $A_i \subset A$, $i \in I$ be subalgebras of A s.t. the family $(A_i)_{i\in I}$ is monotone independent in (A,φ).

We observe $(A_i)_{i\in I}$ is weakly independent in (A,φ), in the sense of [4,11]; remind the weak-independence has the sense below.

Definition 3.1 *Let* (B,ω) *be a quantum probability space and* $B_i \subset B, i \in I$ *be subalgebras. The family* $(B_i)_{i\in I}$ *is weakly independent in* (B,ω), *if* $\omega(x_1...x_n) = \omega(x_1...x_p)\omega(x_{p+1}...x_n)$, *for all* $n > p \geq 1$, *all* $i_j \in I$, *all* $x_j \in B_{i_j}$, *s.t. the sets* $\{i_1,...,i_p\}$ *and* $\{i_{p+1},...,i_n\}$ *are disjoint. If* $B \supset S_i$, $i \in I$ *are subsets, then* $(S_i)_{i\in I}$ *is weakly independent, if* $(B_i)_{i\in I}$ *is weakly independent*; B_i *being the subalgebra of* B *generated by* S_i.□

Lemma 3.2 $(A_i)_{i\in I}$ *is weakly independent in* (A,φ); *i.e.,*
$\varphi(a_1 \cdots a_n) = \varphi(a_1 \cdots a_p)\varphi(a_{p+1} \cdots a_n)$, *for all* $n > p \geq 1$, *all* $i_j \in I$, *all* $a_j \in A_{i_j}$, *s.t. the sets* $\{i_1,...,i_p\}$ *and* $\{i_{p+1},...,i_n\}$ *are disjoint.*

Proof. It suffices to assume $i_1 \neq i_2 \neq ... \neq i_n$. The factorization property by marginal peaks implies the assertion for $n = 2$. For $n = 3$, the assertion follows via Remark 2.2 and the $n = 2$ case.

Let $n > 3$. Suppose the statement true for any $r < n$. We may conclude by induction due to the inferences below.

If $i_1 > i_2$, we get $\varphi(a_1 \cdots a_n) = \varphi(a_1 \cdots a_p)\varphi(a_{p+1} \cdots a_n)$ through the factorization property by marginal peaks (only), for $p = 1$, and in addition the induction hypothesis, for $p \geq 2$.

When $i_1 < i_2$ and $j \mapsto i_j$ has no internal peaks, then this map is strictly increasing on $\{1,...,n\}$; thus, $\varphi(a_1 \cdots a_n) = \varphi(a_1 \cdots a_p)\varphi(a_{p+1} \cdots a_n)$ by Remark 2.2.

Otherwise, consider an internal peak a_k in $a_1 \cdots a_n$. If $p < k-1$ or $p \geq k+1$, the assertion results by the inductive hypothesis and the factorization property by a_k. If $p \in \{k-1, k\}$, the factorization property by a_k again entails
$\varphi(a_1 \cdots a_n) = \varphi(a_k)\varphi(a_1 \cdots a_{k-1}a_{k+1} \cdots a_n) = \varphi(a_1...a_{k-1})\varphi(a_k)\varphi(a_{k+1}...a_n) = \varphi(a_1 \cdots a_p)\varphi(a_{p+1} \cdots a_n)$, via the induction hypothesis and the factorization property by marginal peaks.□

For $w = a_1 \cdots a_n \in A$ s.t. every $a_j \in A_{i_j}$, we say w has a_k as singleton when $i_k \neq i_j$, for any $j \neq k$.

For the next statement see, e.g., [10, Prop. 8.14] and [15, Lemma 6.1].

Lemma 3.3 *Let $w = a_1 \cdots a_n \in A$, s.t. every $a_j \in A_{i_j}$, and w has a centered singleton a_k. Then* $\varphi(w) = 0$.

Proof. We sketch the proof only for the reader's convenience. It suffices to assume w is reduced. If $k \in \{1, n\}$, apply Lemma 3.2 and the centeredness of a_k. It rests to consider $2 \le k \le n-1$. For $n = 3$, the statement results via Lemma 3.2, Remark 2.2 and the centeredness of a_k.

Let $n > 3$. Suppose the statement true for any $a_1 \cdots a_r \in A$ of length $r < n$; check it for $w = a_1 \cdots a_n \in A$, as follows. If $i_{n-1} < i_n$, the factorization property by a_n implies $\varphi(w) = \varphi(a_1 \cdots a_{n-1})\varphi(a_n)$=0, because $\varphi(a_1 \cdots a_{n-1}) = 0$ by the induction hypothesis or Lemma 3.2 and the centeredness of a_k. If $i_{n-1} > i_n$, and $j \mapsto i_j$ has no internal peaks, then this map is strictly decreasing on $\{1, ..., n\}$; so, $\varphi(a_1 \cdots a_k \cdots a_n) = \varphi(a_1) \cdots \varphi(a_k) \cdots \varphi(a_n) = 0$ by the Remark 2.2 and the centeredness of a_k. Alternatively, when the singleton a_k is even a peak in w, then w is a simple random variable in (A, φ), and $\varphi(w) = 0$, due to Remark 2.2. Otherwise, consider an internal peak a_p in w. Then $\varphi(a_1 \cdots a_{p-1} a_{p+1} \cdots a_n)$=0, by the inductive hypothesis. Thus, $\varphi(w) = 0$ via the factorization property by a_p.□

Lemma 3.4 *Let $w = a_1 \cdots a_n \in A$, s.t. all $a_j \in A_{i_j}$ are centered, and the ordered partition π associated to w is a crossing pairing. Then* $\varphi(w) = 0$.

Proof. In view of Lemma 3.2, and Remark 2.2, it remains to consider: π is irreducible, $i_1 < i_2$, $i_{n-1} > i_n$, all internal peaks in the reduced form of w arise from some interval blocks of π ; and, under its reduced form, any subword obtained from w by excluding such an internal peak has no other internal peaks arising from interval blocks of π which does not occur in w.

Let $c_0, c_1, \ldots, c_r$ be all these internal peaks in the reduced form of w; let I_k be the interval block of π corresponding to c_k. The factorization property by every c_k successively implies (with $w_1, \ldots, w_r$ under the reduced form):

$\varphi(w) = \varphi(c_0)\varphi(w_1) = \varphi(c_0)\varphi(c_1)\varphi(w_2) = ... = \varphi(c_0) \cdots \varphi(c_{r-1})\varphi(w_r) = \varphi(c_0) \cdots \varphi(c_r)\varphi(w_0)$; where $w_1, \ldots, w_r$ and w_0 are the obvious subwords of w corresponding to the ordered subpartitions $\pi_1, \ldots, \pi_r$ and π_0 of π, respectively, given by $\pi_1 := \pi \setminus \{I_0\}$, $\pi_j := \pi_{j-1} \setminus \{I_{j-1}\}$, for $j = 2, ..., r$, and $\pi_0 := \pi_r \setminus \{I_r\}$.

Thus, $\pi_1, \ldots, \pi_r$ and π_0 are crossing pairings; and π_0 has no intervals giving internal peaks in w_0. Hence w_0 is a simple random variable in (A, φ); and $\varphi(w_0) = 0$ by Remark 2.2. □

When (A, φ) is a quantum probability space as before, and $x_1, x_2 \in A$ are random variables s.t. one of them is centered w.r.t. φ, then $\varphi(x_1 x_2) = k_2^{\varphi}(x_1, x_2)$; whenever, e.g., k_2^{φ} are the tensor/free/Boolean/monotone cumulants (see, e.g., [2, 22]) w.r.t. φ, respectively, of order two. In the sequel, we may use any of these choices; and we define $k_\pi(x_1, ..., x_n) := \prod_{(i,j), \in B(\pi)} k_2^{\varphi}(x_i, x_j)$, for $\pi \in M_2(n)$ and $x_k \in A$.

We illustrate the last lemma by the following partitions in $ONC_2(n)$ associated to $w = a_1 \cdots a_n \in A$.

Examples 3.5 1) If $n=4$, let $\pi_1=((2,3)_1,(1,4)_2)$ and $\pi_2=((1,4)_1,(2,3)_2)$, which are non-monotone and, respectively, monotone. Their interval gives $a_2a_3=:c_2\in A_{i_2}$. Thus, $w=a_1c_2a_4$ as reduced word. For π_1, we get $\varphi(w)=\varphi(a_1c_2)\varphi(a_4)=0$, because a_4 is a marginal peak in w, and this is a simple random variable. For π_2, remark c_2 is an internal peak in w; so,

$\varphi(w)=\varphi(c_2)\varphi(a_1a_4)=k_2^{\varphi}(a_2,a_3)k_2^{\varphi}(a_1,a_4)=k_{\pi_2}(a_1,...,a_4)$ via the factorization property by c_2.

2) For $n=6$, let π be any of the monotone pairings $((1,6)_1,(2,3)_2,(4,5)_3)$, $((1,6)_1,(4,5)_2,(2,3)_3)$, and $((1,6)_1,(2,5)_2,(3,4)_3)$. For each of them, the interval block $(\cdot,\cdot)_3$ gives the unique internal peak c in the reduced form of w; thus, $\varphi(w)=\varphi(c)\varphi(w')$, via the factorization property by c, where w' is the subword obtained from w by excluding c. The computation of $\varphi(w')$ reduces to the above example for $n=4$, implying $\varphi(w)=k_{\pi}(a_1,...,a_6)$ in each case.

3) For $n=8$, let π be any of the following monotone partitions: $\{(1,8)_1,(2,5)_2,(3,4)_s,(6,7)_t\}$, $\{(1,8)_1,(4,7)_2,(2,3)_s,(5,6)_t\}$, $\{(1,8)_1,(2,7)_2,(3,4)_s,(5,6)_t\}\in NC_2(8)$, with $3\le s,t\le 4$; and $((1,8)_1,(6,7)_2(2,5)_3,(3,4)_4)$, $((1,8)_1,(2,3)_2,(4,7)_3,(5,6)_4)\in ONC_2(8)$. For each of them, the interval block $(\cdot,\cdot)_4$ gives the left or right internal peak c in the reduced form of w; so, $\varphi(w)=\varphi(c)\varphi(w'')$, via the factorization property by c again; where w'' is the subword obtained from w by excluding c. The computation of $\varphi(w'')$ reduces to the examples before for $n=6$. Thus, one finally gets $\varphi(w)=k_{\pi}(a_1,...,a_8)$ always. □

Lemma 3.6 *Let $w=a_1\cdots a_n\in A$, s.t. all $a_j\in A_{i_j}$ are centered, and the ordered partition π associated to w is a non-crossing pairing. Then $\varphi(w)=0$, if π is not monotone; but $\varphi(w)=k_{\pi}(a_1,...,a_n)$, if π is monotone.*

Proof. Due to Lemma 3.2, we may consider $(1,n)_{\cdot}\in\pi$. If, under its reduced form, w has marginal peaks, or $i_1<i_2$, $i_{n-1}>i_n$, and w has an internal peak a_p that does not arise from an interval block of π, then w is a simple random variable in (A,φ); and observe: π has a pair of blocks $(k,\cdot)_{\cdot}\prec(k+1,\cdot)_{\cdot}$, with $k\in\{1,p\}$, or $(l,\cdot)_{\cdot}\prec(l-1,\cdot)_{\cdot}$, with $l\in\{p,n\}$, for which the colouring does not reflect the order. Thus, π is not monotone then, and $\varphi(w)=0$, by Remark 2.2.

Therefore, the assertion being trivial for $n=2$, it remains to consider that $(1,n)_{\cdot}\in\pi$, $i_1<i_2$, $i_{n-1}>i_n$, and w has (under its reduced form) only internal peaks arising from some interval blocks of π. Note that, for any pair of blocks $P_k\prec P_l$ in π involving such an interval block giving the left (right) internal peak in w, the colouring is order-reflecting: $k<l$.

For $n=4$, 6, see Examples 3.5. For $n=8$, there are only the following pairings, besides of the partitions in Examples 3.5: $\{(4,5)_1,(1,8)_2,(2,3)_s,(6,7)_t\}\in NC_2(8)$, with $3\le s,t\le 4$ (non-monotone); $\{(1,8)_1,(2,3)_p,(4,5)_q,(6,7)_r\}\in NC_2(8)$, with $2\le p,q,r\le 4$, and $((1,8)_1,(2,7)_2,(3,6)_3,(4,5)_4)\in ONC_2(8)$ (monotone). For each of these pairings, the interval block $(\cdot,\cdot)_4$ gives the left or right internal peak c in the reduced form of w. Then the subword w' obtained from w by excluding c has a marginal peak in both of these non-monotone cases (when w' is a simple random variable and $\varphi(w')=0$). And for the monotone cases, the computation of $\varphi(w')$ reduces again to the three cases for $n=6$ from Examples 3.5; thus, one finally gets $\varphi(w)=k_{\pi}(a_1,...,a_8)$ always.

Let $n>8$. Suppose the assertion true for all $p<n$. To conclude by induction, remark the next facts.

Let $a_r a_{r+1} =: c_r \in A_{i_r}$ be, for instance, the right internal peak in (the reduced form of) w, arising (as a singleton) in w from an interval $(r, r+1)_\bullet \in \pi$, with $i_{r-1} < i_r = i_{r+1} > i_{r+2}$. Therefore we may express $w = a_1 x c_r y a_n$; where x, y are void or arbitrary products of $a_j \in A_{i_j}$ with $\varphi(a_j) = 0$; but, y (as non-void factor in w) has no internal peaks. After reducing x and y, the factorization property by c_r implies again $\varphi(w) = \varphi(c_r)\varphi(a_1 x y a_n)$. The ordered subpartition of π associated to $a_1 x y a_n$ belongs now to $ONC_2(n-2)$.

If π is not monotone, this ordered subpairing of π is not monotone; hence $\varphi(w) = 0$, by the inductive hypothesis. If π is monotone, this ordered subpairing of π is monotone, too. We may proceed as in [13, Lemma 3.8]. Let $\rho \in M_2(\{2,...,n-1\} \setminus \{r, r+1\})$ be the ordered sub-partition of π associated to xy. Since $\{(1,n)_\bullet\} \amalg \rho =: \sigma \in M_2(\{1,...,n\} \setminus \{r, r+1\})$ is the ordered sub-partition of π associated to $a_1 x y a_n$, the induction hypothesis implies

$$\varphi(a_1 x y a_n) = k_\sigma(a_1, a_2, ..., a_{r-1}, a_{r+2}, ..., a_{n-1}, a_n) = k_2^\varphi(a_1, a_n) k_\rho(a_2, ..., a_{r-1}, a_{r+2}, ..., a_{n-1}).$$

Thus, $\varphi(w) = k_2^\varphi(a_r, a_{r+1}) k_2^\varphi(a_1, a_n) k_\rho(a_2, ..., a_{r-1}, a_{r+2}, ..., a_{n-1}) = k_\pi(a_1, ..., a_n)$; because $\pi = \{(1,n)_\bullet\} \amalg \tau$, with $\tau := \rho \amalg \{(r, r+1)_\bullet\} \in M_2(\{2,...,n-1\})$.□

4. MONOTONE GAUSSIAN FAMILY AND MULTIVARIATE CLT

Let I be an arbitrary index set. We recall a scalar matrix $q = \{q_{ij}\}_{i,j \in I}$ is positive if and only if $\sum_{k,l=1}^{n} q_{i_k, i_l} \overline{\lambda}_k \lambda_l \geq 0$, for all n, all $i_1, ..., i_n \in I$, and all $\lambda_1, ..., \lambda_n \in \mathbb{C}$.

The following definition is inspired from [2, 3, 8, 13 , 22].

Definition 4.1 *Let $q = \{q_{ij}\}_{i,j \in I}$ be a (positive) scalar matrix. Let (A, φ) be a quantum (*-) probability space. A family of (selfadjoint) random variables $g = (g_i)_{i \in I}$ in this is called a centered monotone Gaussian family of covariance q, if its distribution is of the following form, for all $j \in \mathbb{N}$ and all $i_1, ..., i_j \in I$:*

$$\varphi(g_{i_1} ... g_{i_j}) = \sum_{\pi \in M_2(j)} \tfrac{1}{|\pi|!} k_\pi(g_{i_1}, ..., g_{i_j}); \text{ where } k_\pi(g_{i_1}, ..., g_{i_j}) := \prod_{(k,l)_\bullet \in B(\pi)} q_{i_k i_l}. \square$$

Theorem 4.2 *Let (A, φ) be a quantum (*-)probability space, and $\{X_r^i, i \in I\} \subset A$, $r \in \mathbb{N}$ be a sequence of monotone independent sets of (selfadjoint) random variables in this, s.t. $X_r = (X_r^i)_{i \in I}$ has the same joint distribution for all $r \in \mathbb{N}$, and all variables are centered, both w.r.t. φ. Consider, for every $N \geq 1$, the sums $S_N^i := \frac{1}{\sqrt{N}} \sum_{r=1}^{N} X_r^i \in A$, and $S_N := (S_N^i)_{i \in I}$ as random vector in (A, φ). Denote the covariance of the variables w.r.t. φ by $q = \{q_{ij}\}_{i,j \in I}$; i.e., $q_{ij} := \varphi(X_r^i X_r^j)$. Then $S_N \xrightarrow{distr} g$; where $g = (g_i)_{i \in I}$ is a centered monotone Gaussian family of (positive) covariance q.*

Proof. Since all X_r have the same joint distribution w.r.t. φ, and form monotone independent sets, the joint moments of $(X_r)_{r \in N}$ are invariant under order-preserving injective maps, and thus, for all fixed

$j \in \mathbb{N}$ and all $i_1,...,i_j \in I$, the moment $\varphi(X_{r_1}^{i_1}...X_{r_j}^{i_j})$ depends only on the ordered partition $\pi \in OP(j)$ corresponding to $(r_1,...,r_j) \in \mathbb{N}^j$. We may denote $\varphi(X_{r_1}^{i_1}...X_{r_j}^{i_j}) =: \varphi(\pi; i_1,...,i_j)$.

Thus,

$$\varphi(S_N^{i_1}...S_N^{i_j}) = (\tfrac{1}{\sqrt{N}})^j \sum_{r_1,...,r_j=1}^{N} \varphi(X_{r_1}^{i_1}...X_{r_j}^{i_j}) = (\tfrac{1}{\sqrt{N}})^j \sum_{\pi \in OP(j)} C_N^{|\pi|} \varphi(\pi; i_1,...,i_j),$$

as in [13,15]; where $|\pi|$ denotes as before the number of blocks in π; and the number of representatives of the equivalence class corresponding to the involved partition $C_N^{|\pi|} := N!/|\pi|!(N-|\pi|)!$ grows asymptotically like $N^{|\pi|}$ for large N. By Lemma 3.3, every partition with singletons has null contribution in the sum above. But the partitions without singletons have $|\pi| \le \frac{j}{2}$ blocks, and the limit of the factor $(\frac{1}{\sqrt{N}})^j C_N^{|\pi|}$ is 0, if $|\pi| < \frac{j}{2}$; and is $\frac{1}{|\pi|!}$, if $|\pi| = \frac{j}{2}$. So $\lim_{N\to\infty} \varphi(S_N^{i_1}...S_N^{i_j}) = \sum_{\pi \in OP_2(j)} \frac{1}{|\pi|!} \varphi(\pi; i_1,...,i_j)$, because π is a pairing, if $\pi \in OP(j)$ has no singletons and its number of blocks is equal to $\frac{j}{2}$. Thus, the odd moments vanish, since $\pi \in OP_2(j)$ is empty, when j is odd.

We conclude, by Lemmata 3.4 and 3.6, because the crossing ordered pairings or the non-monotone non-crossing ordered pairings have null contribution in the previous sum, and, respectively, the monotone pairings have the claimed contribution.□

Remarks 4.3 1) As in the classical, commutative (:tensor) [10, 22], univariate monotone [10, 24] or (c-)free cases [13,22] (see also [5,27], for short proofs), the assumption of being identically distributed for the involved random vectors may be replaced by the pair (i)&(ii) below, with essentially the same proof as above, but we do not detail this here:

i) $\sup_{r\in\mathbb{N}} |\varphi(X_r^{i_1}...X_r^{i_j})| < \infty$ (for all j, and all $i_1,...,i_j \in I$);

ii) there exist $q_{ij} = \lim_{N\to\infty} \frac{1}{N} \sum_{r=1}^{N} \varphi(X_r^i X_r^j)$.

2) By reversing the order on I, we get the statements corresponding to the multivariate CLT for the anti-monotone random variables (in terms of bottoms instead of peaks) via the anti-monotone partitions.

3) The combinatorial description of the joint moments of a Gaussian family (: multivariate normal distribution) involving all pairings instead of all non-crossing pairings (as a semicircular family [22] in the free probability theory) is often named the Isserlis formula [14] in the classical mathematical statistics or probability theory, and usually the Wick formula in the quantum field theory (see, e.g., [23]). By analogy (see also [3,13]), it seems adequate to name an arcsine family such a(n) (anti-)monotone Gaussian family, and the formula describing its joint moments (see also, [8]) may be interpreted as a(n) (anti-)monotone Isserlis-Wick formula. □

In the same way, we can obtain operator-valued versions of these facts or other generalizations, including the case of the c-monotone or the indented independence, but we do detail this elsewere.

ACKNOWLEDGEMENTS

I am deeply grateful to Acad. Ioan Cuculescu and to Acad. Marius Iosifescu for many valuable discussions on these and related topics and their essential support, which has made my work possible. I am deeply grateful to Acad. Gabriela Marinoschi for many valuable comments and suggestions on all these subjects and moral support. I am deeply grateful to Professor Florin Boca for his interest concerning my work, many stimulating ideas, and moral support. I am deeply grateful to Professor Ioan Stancu-Minasian for his opportune support and much beneficial advice. I am deeply indebted to Professors Vasile Preda, Marius Radulescu and Gheorghita Zbaganu for many stimulating conversations and moral support.

REFERENCES

1. L. ACCARDI, A. BEN GHORBAL, N. OBATA, *Monotone independence, comb graphs, and Bose-Einstein condensation*, Infin. Dim. Anal. Quantum Probab. Rel. Top., **7**, *4*, pp. 419-435, 2004.
2. O. ARIZMENDI, T. HASEBE, F. LEHNER, C. VARGAS, *Relations between cumulants in noncommutative probability*, Adv. in Math., **282,** pp. 56-92., 2015.
3. M. BOZEJKO, R. SPEICHER, *ψ-independent and symmetrized white noises*, in Quantum Probability and Related Topics **VI** (L.Accardi (ed.)) , World Scientific, pp. 219-236, 1992.
4. T. CABANAL-DUVILLARD, V. IONESCU, *Un theoreme central limite pour des variables aleatoires non-commutatives*, C. R. Acad. Sci. Paris, Ser.1, **325**, pp. 1117-1120, 1997.
5. I. CUCULESCU, A.G. OPREA, *Noncommutative Probability*, Kluwer, Dordrecht, 1994.
6. U. FRANZ, *Monotone independence is associative*, Infin. Dim. Anal. Quantum Probab. Rel. Top., **4**, *3*, pp. 401-407, 2001.
7. U. FRANZ, T. HASEBE, S. SCHLEISSINGER, *Monotone increment processes, classical Markov processes and Loewner chains*, Dissertationes Math., **552**, pp.1-119, 2020.
8. T. HASEBE, H. SAIGO, *On operator-valued monotone independence*, Nagoya Math. J., **215** , pp. 151-167, 2014.
9. F. HIAI, D. PETZ, *The Semicircle Law, Free Random Variables, and Entropy*, Math. Surveys and Monographs, Vol. **77** , Amer. Math. Soc., Providence RI, 2000.
10. A. HORA, N. OBATA, *Quantum probability and spectral analysis of graphs*, Theoretical and Mathematical Physics, Springer Science & Business Media, Berlin Heidelberg, 2007.
11. V. IONESCU, *Liberte par rapport a un ensamble arbitraire d'etats*, Preprint IMAR 1995.
12. V. IONESCU, *A note on amalgamated monotone, anti-monotone, and ordered-free products of operator-valued quantum probability spaces*, Rev. Roum. Math. Pures. Appl., 57, 3, pp. 225-243, 2012.
13. V. IONESCU, *A proof of the central limit theorem for c-free quantum random variables*, Proc. Ro. Acad., Series A, **22**, 4, pp. 317-324, 2021.
14. L. ISSERLIS, *On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables*, Biometrika, 12 (1-2), pp.134–139, 1918.
15. R. LENCZEWSKI, R. SALAPATA, *Discrete interpolation between monotone probability and free probability,* Infin. Dim. Anal. Quantum Probab. Rel. Top., **9,** *4*, pp. 77-106, 2006.
16. Y. G. LU, *On the interacting free Fock space and the deformed Wigner law*, Nagoya Math. J., **145** , pp. 1-28, 1997.
17. Y. G. LU, *An interacting free Fock space and the arcsine law*, Probab. Math. Statist., **17**, 1, pp. 149-166, 1997.
18. N. MURAKI, *Noncommutative Brownian motion in monotone Fock space*, Comm. Math. Phys., **183**, pp. 557-570. 1997.
19. N. MURAKI, *Monotonic convolution and monotonic Levy-Hinčin formula*, Preprint 2000.
20. N. MURAKI, *Monotonic independence, monotonic central limit theorem and monotonic law of large numbers*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **4** , pp. 39–58, 2001.
21. N. MURAKI, *The five independences as quasi-universal products,* Infin. Dim. Anal. Quantum Probab. Rel. Top., **5**, 1, pp.113-134, 2002.
22. A. NICA, R. SPEICHER, *Lectures on the Combinatorics of Free Probability*, Cambridge Univ. Press, 2006.
23. M.E. PESKIN, D.V. SCHROEDER, *An Introduction to Quantum Field Theory*, Addison-Wesley, 1995.
24. H. SAIGO, *A simple proof for monotone CLT*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **13**, pp. 339–343, 2010.
25. R. SPEICHER, *On universal products*, in "Free Probability Theory" (D.-V. Voiculescu ed.), Fields Inst. Commun., **12**, pp. 257-266, 1997.
26. D.-V. VOICULESCU, *Free probability theory: random matrices and von Neumann algebras*, Proc. ICM, Birkhauser, Basel, pp. 227-241, 1994.
27. D.-V. VOICULESCU, K. DYKEMA, A. NICA, *Free Random Variables*, C.R.M. Monograph Series, No. **1**, Amer. Math. Soc., Providence RI, 1992.

Received February 5, 2023