The Analysis of Some Classes of Nonlinear PDEs

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Habilitation Thesis

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1.2. The set of eigenvalues of a problem involving Neumann boundary condition

Assume $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$.

We consider the eigenvalue problem

$$\begin{cases} -\Delta_p u - \Delta u = \lambda u & \text{in } \Omega, \\ \left(|\nabla u|_N^{p-2} + 1\right) \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

when $p \in (1, \infty) \setminus \{2\}$ is a real number, $\Delta_p u := \operatorname{div}(|\nabla u|_N^{p-2} \nabla u)$ stands for the p-Laplace operator and ν denotes the outward unit normal to $\partial\Omega$.

Background

We consider the problem

$$\begin{cases} -\Delta\, u = \lambda u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where ν denotes the outward unit normal to $\partial\Omega$.

Problem (2) possesses an unbounded sequence of eigenvalues, more precisely

$$0 = \lambda_0^N < \lambda_1^N < \lambda_2^N \leq \cdots \leq \lambda_n^N \leq \cdots .$$

Consequently, in this case the set of eigenvalues of problem (2) is discrete.

The first positive eigenvalue of problem (2) is

$$\lambda_1^N := \inf_{u \in W^{1,2}(\Omega)\setminus\{0\},\ \int_\Omega u\, dx = 0} \frac{\displaystyle\int_\Omega |\nabla u|_N^2\, dx}{\displaystyle\int_\Omega u^2\, dx} . \tag{3}$$

Continuous spectrum

We consider the problem

$$\begin{cases} -\Delta_p u = \lambda u & \text{in } \Omega, \\ |\nabla u|_N^{p-2} \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (4)$$

with either $p \in \left(\left(\frac{2N}{N+2}, \infty\right) \setminus \{2\}\right) \cap (1, N)$ or $p > N$.

Definition

$\lambda \in \mathbb{R}$ is an eigenvalue of problem (4), if $\exists\, u \in W^{1,p}(\Omega) \setminus \{0\}$ s.t.

$$\int_\Omega |\nabla u|_N^{p-2} \nabla u \nabla \varphi \, dx = \lambda \int_\Omega u\varphi \, dx, \quad \forall\, \varphi \in W^{1,p}(\Omega)\,. \quad (5)$$

Theorem

The set of eigenvalues of problem (4) is the interval $[0, \infty)$.

Main problem

We consider the problem

$$\begin{cases} -\Delta_p u - \Delta u = \lambda u & \text{in } \Omega, \\ \left(|\nabla u|_N^{p-2} + 1\right) \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $p \in (1, \infty)\setminus\{2\}$ is a real number.

Definition 1.1

The parameter $\lambda \in \mathbb{R}$ is an eigenvalue of problem (6) if there exists $u \in W^{1,\max\{2,p\}}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|_N^{p-2} \nabla u \nabla \varphi \, dx + \int_\Omega \nabla u \nabla \varphi \, dx - \lambda \int_\Omega u\varphi \, dx = 0, \tag{7}$$

for all $\varphi \in W^{1,\max\{2,p\}}(\Omega)$.

Background

Theorem (M. Mihăilescu, CPAA, 2011)

If $p \in (2, \infty)$, the set of eigenvalues of problem (6) is given by

$$\{0\} \cup (\lambda_1(p), \infty),$$

where

$$\lambda_1(p) := \inf_{u \in W^{1,p}(\Omega)\setminus\{0\},\ \int_\Omega u\, dx = 0} \frac{\int_\Omega |\nabla u|_N^2\, dx}{\int_\Omega u^2\, dx} > 0. \tag{8}$$

M. Mihăilescu (2011),
An eigenvalue problem possessing a continuous family of eigenvalues plus an isolated eigenvalue,
Comm. on Pure and Applied Analysis **10**, 701–708.

Spectrum consisting in a continuous part plus an isolated point

The main result on problem (6) in the case $p \in (1,2)$ is given by the following theorem.

Theorem (M. Fărcășeanu, M. Mihăilescu, D. S-D, NA, 2015)

If $p \in (1,2)$, the set of eigenvalues of problem (6) is given by

$$\{0\} \cup (\lambda_1^N, \infty),$$

where

$$\lambda_1^N := \inf_{u \in W^{1,2}(\Omega)\setminus\{0\},\ \int_\Omega u\,dx = 0} \frac{\int_\Omega |\nabla u|_N^2\,dx}{\int_\Omega u^2\,dx} > 0.$$

Case $p \in (1,2)$

Definition

$\lambda \in \mathbb{R}$ is an eigenvalue of problem (6) if there exists $u \in W^{1,2}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|_N^{p-2} \nabla u \nabla \varphi \, dx + \int_\Omega \nabla u \nabla \varphi \, dx - \lambda \int_\Omega u \varphi \, dx = 0, \qquad (7)$$

for all $\varphi \in W^{1,2}(\Omega)$.

Case $p \in (1,2)$

Definition

$\lambda \in \mathbb{R}$ is an eigenvalue of problem (6) if there exists $u \in W^{1,2}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|_N^{p-2} \nabla u \nabla \varphi \, dx + \int_\Omega \nabla u \nabla \varphi \, dx - \lambda \int_\Omega u \varphi \, dx = 0, \qquad (7)$$

for all $\varphi \in W^{1,2}(\Omega)$.

In order to go further, we define

$$V_2 := \{u \in W^{1,2}(\Omega); \int_\Omega u \, dx = 0\}.$$

We recall that

$$W^{1,2}(\Omega) = V_2 \oplus \mathbb{R}.$$

Lemma 1.2.4.

Every $\lambda \in (\lambda_1^N, \infty)$ is an eigenvalue of problem (6).

For each $\lambda \in (\lambda_1^N, \infty)$ we define the functional $I_\lambda : V_2 \to \mathbb{R}$ by

$$I_\lambda(u) := \frac{1}{2}\int_\Omega |\nabla u|_N^2 \, dx + \frac{1}{p}\int_\Omega |\nabla u|_N^p \, dx - \frac{\lambda}{2}\int_\Omega u^2 \, dx. \tag{9}$$

It is standard to prove that $I_\lambda \in C^1(V_2 \backslash \{0\}, \mathbb{R})$ with the derivative given by

$$\langle I'_\lambda(u), \varphi\rangle = \int_\Omega \nabla u \nabla \varphi \, dx + \int_\Omega |\nabla u|_N^{p-2} \nabla u \nabla \varphi \, dx - \lambda \int_\Omega u\varphi \, dx, \tag{10}$$

$\forall\ u \in V_2 \backslash \{0\}$, $\forall\ \varphi \in V_2$.

Thus, λ is an eigenvalue of problem (6) if and only if I_λ possesses a nontrivial critical point.

We define the so-called Nehari manifold by

$$N_\lambda := \{u \in V_2 \backslash \{0\} : \langle I'_\lambda(u), u\rangle = 0\}$$

$$= \left\{u \in V_2 \backslash \{0\} : \int_\Omega |\nabla u|_N^2 \, dx + \int_\Omega |\nabla u|_N^p \, dx = \lambda \int_\Omega u^2 \, dx\right\}.$$

On N_λ functional I_λ has the following expression

$$I_\lambda(u) = \left(\frac{1}{p} - \frac{1}{2}\right) \int_\Omega |\nabla u|_N^p \, dx.$$

Set $m := \inf_{u \in N_\lambda} I_\lambda(u) \geq 0$.

We proceed in 5 steps:

Step 1. $N_\lambda \neq \emptyset$.
Step 2. Every minimizing sequence for I_λ on N_λ is bounded.
Step 3. $m > 0$.
Step 4. $\exists\, u \in N_\lambda$ s.t. $I_\lambda(u) = m$.
Step 5. u found on step 4 is a critical point for I_λ.

2. Chapter 2: The asymptotic behavior of solutions for some classes of PDEs
    2.1 The limiting bevahior of solutions for a class of problems involving the p-Laplace operator and an exponential term
    2.2 Convergence of the sequence of solutions for a family of eigenvalue problems
    2.3 A limiting problem for a family of eigenvalue problems involving p-Laplacians
    2.4 The asymptotic behavior of solutions to a class of inhomogeneous problems
    2.5 The limiting behavior of solutions to inhomogeneous eigenvalue problems in Orlicz-Sobolev spaces
    2.6 The asymptotic behavior of a class of φ-harmonic functions

2.1. A class of problems involving the p-Laplace operator and an exponential term

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary, $\partial\Omega$.

For each $p > N$, consider the problem

$$\begin{cases} -\Delta_p u = \lambda e^u, & \text{in } \ \Omega, \\ u = 0, & \text{on } \ \partial\Omega, \end{cases} \tag{11}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|_N^{p-2} \nabla u)$.

Definition

We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of problem (11),

$$\int_\Omega |\nabla u|_N^{p-2} \nabla u \nabla \varphi \, dx = \lambda \int_\Omega e^u \varphi \, dx, \quad \forall \, \varphi \in W_0^{1,p}(\Omega)\,. \tag{12}$$

Theorem (Morrey's inequality)

For each $p > N$, there exists a positive constant C_p such that

$$\|u\|_{L^\infty(\Omega)} \leq C_p \||\nabla u|_N\|_{L^p(\Omega)}, \quad \forall\, u \in W_0^{1,p}(\Omega)\,, \tag{13}$$

where

$$C_p = p|B_1(0)|^{-\frac{1}{p}} N^{-\frac{N(p+1)}{p^2}} (p-1)^{\frac{N(p-1)}{p^2}} (p-N)^{\frac{N-p^2}{p^2}} \lambda_1(p)^{\frac{N-p}{p^2}}$$

and

$$\lambda_1(p) := \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_N^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}.$$

Furthermore, it is known that

$$\lim_{p\to\infty} C_p = \|\mathrm{dist}(\cdot, \partial\Omega)\|_{L^\infty(\Omega)},$$

where $\mathrm{dist}(x, \partial\Omega) := \inf_{y\in\partial\Omega} |x - y|_N, \ \ \forall\, x \in \Omega.$

F. Charro & E. Parini
Limits as $p \to \infty$ of p-Laplacian problems with a superdiffusive power-type nonlinearity: Positive and sign-changing solutions,
J. Math. Anal. Appl. **372** (2010), 629-644.

Background

Theorem

For each $p > N$, there exists a positive real number λ_p such that for each $\lambda \in (0, \lambda_p)$, problem

$$\begin{cases} -\Delta_p u = \lambda e^u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

has a weak solution.

J. A. Aguilar Crespo and I. Peral Alonso,
On an elliptic equation with exponential growth,
Rend. Sem. Mat. Univ. Padova **96** (1996), 143–175.

Main results

Theorem 2.1.1 [M. Mihăilescu, D.S.-D., C. Varga, ESAIM COCV, 2018]

There exists a positive real number λ^* such that for each $\lambda \in (0, \lambda^*)$ and for each $p > N$, problem (11), namely

$$\begin{cases} -\Delta_p u = \lambda e^u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

has a nonnegative weak solution.

Theorem 2.1.2 [M. Mihăilescu, D.S.-D., C. Varga, ESAIM COCV, 2018]

For each $\lambda \in (0, \lambda^\star)$, let u_p be the nonnegative solution of problem (11), given by Theorem 1.2.1. Then $\{u_p\}$ converges uniformly in Ω to $\operatorname{dist}(\cdot, \partial\Omega)$ as $p \to \infty$.

Background

$$\begin{cases} -\Delta_p u = f(x), & \text{for } x \in \Omega, \\ u = 0, & \text{for } x \in \partial\Omega, \end{cases} \tag{14}$$

where $f \in L^\infty(\Omega) \setminus \{0\}$ is a positive function.

- T. Bhattacharya, E. DiBenedetto, & J. Manfredi, *Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems,* *Rend. Sem. Mat. Univ. Politec. Torino* (1991), 15-68.
- B. Kawohl, *On a family of torsional creep problems,* *J. Reine Angew. Math.* **410** (1990), 1-22.
- M. Perez-Llanos & J. D. Rossi, *The limit as $p(x) \to \infty$ of solutions to the inhomogeneous Dirichlet problem of $p(x)$-Laplacian,* *Nonlinear Analysis* **73** (2010), 2027-2035.
- M. Bocea & M. Mihăilescu, *On a family of inhomogeneous torsional creep problems,* *Proceedings of the American Mathematical Society* **145** (2017), 4397-4409.

Background

A subsequence of the solutions $u_p > 0$ of the family of problems

$$\begin{cases} -\Delta_p u = \lambda_1(p)|u|^{p-2}u, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{15}$$

converges uniformly in Ω to a nontrivial and nonnegative viscosity solution of the limiting problem

$$\begin{cases} \min\left\{|\nabla u|_N - \frac{u}{\|\mathrm{dist}(\cdot,\partial\Omega)\|_{L^\infty}}, -\Delta_\infty u\right\} = 0 & \text{in} \ \ \Omega, \\ u = 0 & \text{on} \ \ \partial\Omega. \end{cases} \tag{16}$$

P. Juutinen, P. Lindqvist & J. J. Manfredi (1999):
The ∞-eigenvalue problem,
Arch. Rational Mech. Anal. **148**, 89-105.

Main results

Theorem 2.1.1 [M. Mihăilescu, D.S.-D., C. Varga, ESAIM COCV, 2018]

There exists a positive real number λ^* such that for each $\lambda \in (0, \lambda^*)$ and for each $p > N$, problem (11), namely

$$\begin{cases} -\Delta_p u = \lambda e^u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

has a nonnegative weak solution.

Existence of solutions

For each $\lambda > 0$, we introduce the Euler-Lagrange functional associated to problem (11), i.e. $J_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$J_\lambda(u) := \frac{1}{p} \int_\Omega |\nabla u|_N^p \, dx - \lambda \int_\Omega e^u \, dx, \quad \forall \, u \in W_0^{1,p}(\Omega) \, .$$

It is standard to show that $J_\lambda \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ and

$$\langle J_\lambda'(u), \phi \rangle = \int_\Omega |\nabla u|_N^{p-2} \nabla u \nabla \phi \, dx - \lambda \int_\Omega e^u \phi \, dx,$$

for all $u, \ \phi \in W_0^{1,p}(\Omega)$.

Remark: Note that the Direct Method in the Calculus of Variations can not be applied in this case since J_λ fails to be coercive.

Lemma 2.1.1

For each $\lambda \in (0, \lambda_p^\star)$, we have

$$J_\lambda(u) \geq \frac{1}{2}, \quad \forall\, u \in W_0^{1,p}(\Omega) \text{ with } \| |\nabla u|_N \|_{L^p(\Omega)} = p^{1/p}\,,$$

where

$$\lambda_p^\star := \frac{1}{2|\Omega| e^{C_p p^{1/p}}}$$

Lemma 2.1.1

For each $\lambda \in (0, \lambda_p^\star)$, we have

$$J_\lambda(u) \geq \frac{1}{2}, \quad \forall\, u \in W_0^{1,p}(\Omega) \text{ with } \||\nabla u|_N\|_{L^p(\Omega)} = p^{1/p},$$

where

$$\lambda_p^\star := \frac{1}{2|\Omega| e^{C_p p^{1/p}}}$$

and

$$C_p = p|B_1(0)|^{-\frac{1}{p}} N^{-\frac{N(p+1)}{p^2}} (p-1)^{\frac{N(p-1)}{p^2}} (p-N)^{\frac{N-p^2}{p^2}} \lambda_1(p)^{\frac{N-p}{p^2}}.$$

Lemma 2.1.1

For each $\lambda \in (0, \lambda_p^\star)$, we have

$$J_\lambda(u) \geq \frac{1}{2}, \quad \forall\, u \in W_0^{1,p}(\Omega) \text{ with } \||\nabla u|_N\|_{L^p(\Omega)} = p^{1/p},$$

where

$$\lambda_p^\star := \frac{1}{2|\Omega| e^{C_p p^{1/p}}}$$

and

$$C_p = p|B_1(0)|^{-\frac{1}{p}} N^{-\frac{N(p+1)}{p^2}} (p-1)^{\frac{N(p-1)}{p^2}} (p-N)^{\frac{N-p^2}{p^2}} \lambda_1(p)^{\frac{N-p}{p^2}}.$$

Since $\lim_{p\to\infty} C_p = \|\mathrm{dist}(\cdot, \partial\Omega)\|_{L^\infty(\Omega)}$ and $\lim_{p\to\infty} p^{1/p} = 1$ it follows that

$$\lim_{p\to\infty} \lambda_p^\star = \frac{1}{2|\Omega| e^{\|\mathrm{dist}(\cdot,\partial\Omega)\|_{L^\infty(\Omega)}}} > 0.$$

Consequently, defining

$$\lambda^\star := \inf_{p>N} \lambda_p^\star\,, \tag{17}$$

and taking into account that function

$$(1,\infty) \ni p \longrightarrow \lambda_1(p) := \inf_{u\in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_N^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}$$

is continuous we deduce that

$$\lambda_p^\star \geq \lambda^\star > 0, \quad \forall\, p > N\,.$$

Variational solution

Theorem 2.1.3

For each $\lambda \in (0, \lambda^\star)$ and each $p > N$, problem (11) has a nonnegative solution $u_p \in B := B_{p^{1/p}}(0)$ which is characterized by

$$J_\lambda(u_p) = \inf_{\overline{B}} J_\lambda.$$

The proof of this theorem relies on Ekeland's Variational Principle.

Lemma 2.1.2

Fix $\lambda \in (0, \lambda^\star)$ and let u_p be the nonnegative solution of problem (11) given by Theorem 2.1.1. Then there is a subsequence $\{u_p\}$ which converges uniformly in Ω, as $p \to \infty$, to some function $u_\infty \in C(\overline{\Omega})$ with $u_\infty \geq 0$ in Ω.

Proof: Fix $q > N$. For each $p > q$ we have

$$\begin{aligned} \int_\Omega |\nabla u_p|_N^q \, dx &\leq \left(\int_\Omega |\nabla u_p|_N^p \, dx \right)^{q/p} |\Omega|^{1-q/p} \leq p^{q/p} |\Omega|^{1-q/p} \\ &\leq (e^{1/e})^q |\Omega|^{1-q/p} \leq (e^{1/e})^q (1 + |\Omega|) \,. \end{aligned}$$

Thus, $\{|\nabla u_p|_N\}_p$ is uniformly bounded in $L^q(\Omega)$.
It follows that there exists a subsequence (not relabeled) of $\{u_p\}$ and a function $u_\infty \in C(\overline{\Omega})$ such that $u_p \rightharpoonup u_\infty$ weakly in $W_0^{1,q}(\Omega)$ with $q > N$ and $u_p \to u_\infty$ uniformly in Ω.
Moreover, $u_p \geq 0 \;\text{ in }\; \Omega, \;\; \forall \;\; p > N \implies u_\infty \geq 0$ in Ω.

We assume for a moment that the solutions u_p of (11) are sufficiently smooth so that we can perform the differentiation in the PDE

$$-\Delta_p u_p := -\mathrm{div}(|\nabla u_p|_N^{p-2}\nabla u_p) = \lambda e^{u_p} \quad \text{in } \Omega,$$

we get

$$-|\nabla u_p|_N^{p-2}\Delta u_p - (p-2)|\nabla u_p|_N^{p-4}\Delta_\infty u_p = \lambda e^{u_p} \quad \text{in } \Omega, \qquad (18)$$

where $\Delta u := \mathrm{Trace}(D^2 u) = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$ and

$$\Delta_\infty u := \langle D^2 u \nabla u, \nabla u\rangle = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Note that (18) can be rewritten as

$$H_p(u_p, \nabla u_p, D^2 u_p) = 0 \ \text{ in } \ \Omega,$$

$$H_p(y,z,S) := -|z|_N^{p-2}\mathrm{Trace}\, S - (p-2)|z_N|^{p-4}\langle Sz, z\rangle - \lambda e^y,$$

where $y \in \mathbb{R}$, $z \in \mathbb{R}^N$ and S is a real symmetric matrix in $\mathbb{M}^{N\times N}$.

Definition of viscosity solution

We give the definition of viscosity solutions for problem

$$\begin{cases} H_p(u, \nabla u, D^2 u) = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{19}$$

Definition (M. G. Crandal, H. Ishii, P. L. Lions, BAMS, 1992)

An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is called a viscosity subsolution of (19) if $u|_{\partial\Omega} \leq 0$ and, whenever $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ are such that $u(x_0) = \Psi(x_0)$ and $u(x) < \Psi(x)$ if $x \in B(x_0, r) \setminus \{x_0\}$ for some $r > 0$, then $H_p(\Psi(x_0), \nabla\Psi(x_0), D^2\Psi(x_0)) \leq 0$.

A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is called a viscosity supersolution of (19) if $u|_{\partial\Omega} \geq 0$ and, whenever $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ are such that $u(x_0) = \Psi(x_0)$ and $u(x) > \Psi(x)$ if $x \in B(x_0, r) \setminus \{x_0\}$ for some $r > 0$, then $H_p(\Psi(x_0), \nabla\Psi(x_0), D^2\Psi(x_0)) \geq 0$.

Lemma 2.1.3

A continuous weak solution of (11) is also a viscosity solution of (11).

Next, we compute the limit of

$$H_p(u_p, \nabla u_p, D^2 u_p) = 0 \ \text{ in } \ \Omega$$

as $p \to \infty$. More exactly, we consider the sequence of viscosity solutions $\{u_p\}$ and we would like to find out what equation is satisfied by any cluster point of this sequence, which is u_∞.

Limit function

Theorem 2.1.3

Let u_∞ be the function obtained as a uniform limit of a subsequence of $\{u_p\}$ in Lemma 2.1.2. Then u_∞ is a viscosity solution of problem

$$\begin{cases} \min\{|\nabla u|_N - 1, -\Delta_\infty u\} = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{20}$$

It is well-known that equation (20) has as unique solution $\mathrm{dist}(\cdot, \partial\Omega)$, namely the distance function to the boundary of Ω. The entire sequence u_p converges uniformly to $\mathrm{dist}(\cdot, \partial\Omega)$ in Ω, as $p \to \infty$.

R. Jensen,
Uniqueness of Lipschitz Extensions: Minimizing the Sup Norm of the Gradient,
Arch. Rational Mech. Anal. **123** (1993), 51-74.

3.1. Anisotropic Torsional Creep Problem

“Torsional creep”

the permanent plastic deformation of a material subject to a torsional moment for an extended period of time and at sufficiently high temperature.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be an open, bounded domain with smooth boundary, $\partial\Omega$.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be an open, bounded domain with smooth boundary, $\partial\Omega$.

For each real number $p > 1$, torsional creep problems are modelled by the family of equations

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{21}$$

which possess unique solutions denoted by $u_p \in W_0^{1,p}(\Omega)$ (actually, $u_p \in C^{1,\alpha}(\overline{\Omega})$).

L. M. Kachanov: The theory of creep, Nat. Lending Lib. for Science and Technology, Boston Spa, Yorkshire, England, 1967.

L. M. Kachanov: Foundations of the theory of plasticity, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1971.

The limit problem of the above family of equations, as $p \to \infty$, is given by

$$\begin{cases} \min\{|\nabla u|_N - 1, -\Delta_\infty u\} = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{22}$$

and it possesses as unique solution the distance function to the boundary of Ω, i.e.

$$\operatorname{dist}(x, \partial\Omega) := \inf_{y \in \partial\Omega} |x - y|_N, \quad \forall\, x \in \Omega\,.$$

Problem (22) models the *perfect plastic torsion*.

- Payne & Philippin (1977) showed that

$$\lim_{p\to\infty}\int_\Omega u_p(x)dx = \int_\Omega \text{dist}(x,\partial\Omega)dx.$$

L. E. Payne & G. A. Philippin (1977):
Some applications of the maximum principle in the problem of torsional creep,
SIAM J. Appl. Math. **33**, 446–455.

• Bhattacharya, DiBenedetto & Manfredi (1991) and Kawohl (1990) established the uniform convergence of u_p to $\mathrm{dist}(\cdot, \partial\Omega)$ in $\overline{\Omega}$.

T. Bhattacharya, E. DiBenedetto, & J. Manfredi (1991):
Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems,
Rend. Sem. Mat. Univ. Politec. Torino, special issue, 15–68.

B. Kawohl (1990):
On a family of torsional creep problems,
J. Reine Angew. Math. **410**, 1–22.

Similar Results on the Topic

- Perez-Llanos & Rossi (2010) investigated the family of equations

$$\begin{cases} -\text{div}(|\nabla u|_N^{p_n(x)-2} \nabla u) = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{23}$$

when $p_n(\cdot)$ is a sequence of continuous functions over Ω which diverges uniformly to infinity in Ω, as $n \to \infty$.

M. Pérez-Llanos & J. D. Rossi (2010):
The limit as $p(x) \to \infty$ of solutions to the inhomogeneous Dirichlet problem of $p(x)$-Laplacian,
Nonlinear Analysis T.M.A. **73**, 2027–2035.

Bocea & Mihăilescu (2017) studied the family of problems

$$\begin{cases} -\mathrm{div}\left(\frac{\varphi_n(|\nabla u|_N)}{\varphi_n(1)|\nabla u|_N}\nabla u\right) = 1 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{24}$$

where $\varphi_n \in C^1(\mathbb{R}, \mathbb{R})$ are odd, increasing homeomorphisms s. t.

$$1 < \varphi_n^- \leq \frac{t\varphi_n(t)}{\Phi_n(t)} \leq \varphi_n^+ < \infty, \quad \forall\, t \geq 0 \tag{25}$$

for some constants φ_n^- and φ_n^+ with $1 < \varphi_n^- \leq \varphi_n^+ < \infty$,

$$\varphi_n^- \to \infty \text{ as } n \to \infty, \tag{26}$$

$$\exists\ \beta > 1 \text{ such that } \varphi_n^+ \leq \beta\varphi_n^- \text{ for all } n > 1, \tag{27}$$

where $\Phi_n(t) := \int_0^t \varphi_n(s)ds$.

M. Bocea & M. Mihăilescu (2017):
On a family of inhomogeneous torsional creep problems,
Proc. Amer. Math. Soc. **145**, 4397-4409.

Examples

Examples of functions $\varphi_n : \mathbb{R} \to \mathbb{R}$ which are odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, and for which (25)-(27) hold. For more details, the reader is referred to [Clément *et al.*, Examples 1-3, page 243].

1. $\varphi_n(t) = |t|^{n-2}t$, $n > 1$. We have $\varphi_n^- = \varphi_n^+ = n$;
2. $\varphi_n(t) = \log(1 + |t|^m)|t|^{n-2}t$, $n, m > 1$. Thus, $\varphi_n^- = n$, $\varphi_n^+ = n + m$;
3. $\varphi_n(t) = \frac{|t|^{n-2}t}{\log(1+|t|)}$ if $t \neq 0$, $\varphi_n(0) = 0$, $n > 2$. In this case it turns out that $\varphi_n^- = n - 1$, $\varphi_n^+ = n$.

Ph. Clément, B. de Pagter, G. Sweers & F. de Thélin (2004): Existence of solutions to a semilinear elliptic system through Orlicz-Sobolev spaces, *Mediterr. J. Math.* **1**, 241–267.

• Fărcăşeanu & Mihăilescu (2019) studied the family of problems (24) with

$$\varphi_n(t) := p_n|t|^{p_n-2}te^{|t|^{p_n}}$$

when $p_n \to \infty$.

We have

$$\varphi_n^- := \inf_{t>0} \frac{t\varphi_n(t)}{\Phi_n(t)} = p_n \text{ and } \varphi_n^+ := \sup_{t>0} \frac{t\varphi_n(t)}{\Phi_n(t)} = \infty.$$

M. Fărcăşeanu & M. Mihăilescu (2019):
On a family of torsional creep problems involving rapidly growing operators in divergence form,
Proceedings of the Royal Society of Edinburgh Section A: Mathematics **149**, 495-510.

Framework

Let L, M and N be three positive integers s.t. $L + M = N$.

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- $\forall\, \xi \in \mathbb{R}^N$ we write $\xi = (x, y) \in \mathbb{R}^L \times \mathbb{R}^M$ with

$$x = (x_1, ..., x_L) \in \mathbb{R}^L \text{ and } y = (y_1, ..., y_M) \in \mathbb{R}^M.$$

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Denote by $|\cdot|_L$, $|\cdot|_M$ and $|\cdot|_N$, the Euclidean norms in $\mathbb{R}^L$, $\mathbb{R}^M$ and $\mathbb{R}^N$, respectively.

For $\xi_1 = (\overline{x}, \overline{y}) \in \mathbb{R}^N$ and $\xi_2 = (\tilde{x}, \tilde{y}) \in \mathbb{R}^N$ with $\overline{x},\ \tilde{x} \in \mathbb{R}^L$ and $\overline{y},\ \tilde{y} \in \mathbb{R}^M$ we define the "anisotropic Euclidean distance" on $\mathbb{R}^N$ as

$$d_N(\xi_1, \xi_2) := |\overline{x} - \tilde{x}|_L + |\overline{y} - \tilde{y}|_M \,. \tag{28}$$

For $u : \Omega \to \mathbb{R}$ smooth enough we will use the following notations

$$\nabla_x u := \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_L}\right), \nabla_y u := \left(\frac{\partial u}{\partial y_1}, ..., \frac{\partial u}{\partial y_M}\right), \nabla u := (\nabla_x u, \nabla_y u).$$

Our Problem

For each positive integer n let p_n and q_n be two sequences satisfying $2 \leq p_n \leq q_n < \infty$. Define the functions

$$\varphi_n(t) := p_n|t|^{p_n-2}te^{|t|^{p_n}} \text{ and } \psi_n(t) := q_n|t|^{q_n-2}te^{|t|^{q_n}}.$$

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Next, for a given continuous function $f : \overline{\Omega} \to (0, \infty)$ consider the family of anisotropic problems

$$\begin{cases} -\text{div}_x \left(\frac{\varphi_n(|\nabla_x u|_L)}{\varphi_n(1)|\nabla_x u|_L} \nabla_x u \right) - \text{div}_y \left(\frac{\psi_n(|\nabla_y u|_M)}{\psi_n(1)|\nabla_y u|_M} \nabla_y u \right) = f, & \Omega, \\ u = 0, & \partial\Omega. \end{cases} \tag{29}$$

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Our goal is to study the asymptotic behaviour of the solutions for the family of problems (29) as $n \to \infty$ (provided that $\lim_{n\to\infty} p_n = \infty$).

Problem (29), due to its anisotropic nature, could represent a torsion that twists the material depending on the direction of the variables.

Main results

Theorem 3.1.1 [D. S.-D., NA-RWA, 2020]

For each integer $n \geq 1$, problem (29) has a unique (variational) solution which is nonnegative in Ω, say v_n.

Theorem 3.1.2 [D. S.-D., NA-RWA, 2020]

If

$$\lim_{n\to\infty} p_n = \infty \text{ and } \limsup_{n\to\infty} \frac{\ln(q_n)}{p_n} < \infty,$$

the sequence $\{v_n\}$ converges uniformly in Ω to $\mathrm{dist_a}(\cdot, \partial\Omega)$, where $\mathrm{dist_a}(\cdot, \partial\Omega) : \Omega \to [0, \infty)$ is defined by

$$\mathrm{dist_a}(\xi, \partial\Omega) := \inf_{\eta\in\partial\Omega} \left[|x_\xi - x_\eta|_L + |y_\xi - y_\eta|_M\right], \quad \forall \ \xi \in \Omega.$$

Orlicz-Sobolev space

We define the *anisotropic Orlicz-Sobolev space*

$$W^{1,\Phi_n,\Psi_n}(\Omega) := \left\{u \in L^{\Psi_n}(\Omega) : |\nabla_x u|_L \in L^{\Phi_n}(\Omega),\ |\nabla_y u|_M \in L^{\Psi_n}(\Omega)\right\}$$

endowed with the norm $\|\cdot\|_{1,\Phi_n,\Psi_n}$, where $L^{\Phi_n}(\Omega)$ and $L^{\Psi_n}(\Omega)$ are Orlicz spaces coresponding to $\Phi_n, \Psi_n : \mathbb{R} \to \mathbb{R}$,

$$\Phi_n(t) := \int_0^t \varphi_n(s)\, ds = e^{|t|^{p_n}} - 1$$

and

$$\Psi_n(t) := \int_0^t \psi_n(s)\, ds = e^{|t|^{q_n}} - 1.$$

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- $\left(W^{1,\Phi_n,\Psi_n}(\Omega), \|\cdot\|_{1,\Phi_n,\Psi_n}\right)$ is a Banach space which is not reflexive, but is the dual of a separable Banach space.

We introduce

$$W_n := W^{1,\Phi_n,\Psi_n}(\Omega) \cap \left(\cap_{s>1} W_0^{1,s}(\Omega)\right).$$

- W_n is a linear closed subspace of $W^{1,\Phi_n,\Psi_n}(\Omega)$.
- if $\{u_n\}$ is a bounded sequence in $W^{1,\Phi_n,\Psi_n}(\Omega)$, then it contains a subsequence which converges in the sense of the weak* topology to some $u \in W_n$.

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The Euler-Lagrange functional associated to the problem (29) is $J_n : W_n \to \mathbb{R}$ defined by

$$\begin{aligned} J_n(v) \quad := \quad & \frac{1}{\varphi_n(1)} \int_\Omega \Phi_n(|\nabla_x v(\xi)|_L)\, d\xi + \frac{1}{\psi_n(1)} \int_\Omega \Psi_n(|\nabla_y v(\xi)|_M)\, d\xi \\ & - \int_\Omega f(\xi) v(\xi)\, d\xi. \end{aligned}$$

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- $I_n \notin C^1(W_n, \mathbb{R})$, but it is convex, weakly* lower semicontinuous and coercive. On the other hand, $K \in C^1(W_n, \mathbb{R})$ and

$$\langle K'(v), w \rangle = - \int_\Omega f(\xi) w(\xi)\, d\xi, \quad \forall\, v,\ w \in W_n\,.$$

Thus, following Szulkin, we will work with the following reformulation of problem (29) as a variational inequality

$$\begin{cases} I_n(w) - I_n(v_n) + \langle K'(v_n), w - v_n \rangle \geq 0 \quad \forall\, w \in W_n, \\ v_n \in W_n\,. \end{cases} \tag{30}$$

- $v_n \in W_n$ solving problem (30) will be called a *variational solution* of problem (29).

A. Szulkin (1986):
Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems,
Ann. Inst. H. Poincaré Anal. Non Linéaire **3**, 77–109.

Theorem 3.1.1 [D. S.-D., NA-RWA, 2020]

For each integer $n \geq 1$ such that $2 \leq p_n \leq q_n$, problem (29) has a unique nonnegative variational solution.

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Proof (Existence):

1.
$$\lim_{\substack{\|v\|_{1,\Phi_n,\Psi_n} \to \infty, \\ v \in W_n}} J_n(v) = +\infty.$$

2.
$$-\infty < c_n := \inf_{w \in W_n} J_n(w).$$

3.
$$\exists\ v_n \in W_n \text{ such that } J_n(v_n) = c_n\,.$$

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3. $$\exists\ v_n \in W_n \text{ such that } J_n(v_n) = c_n\ .$$

Nonnegativity: Note that since $J_n(v_n) \geq J_n(|v_n|)$ and J_n possesses a unique critical point, we must have $v_n = |v_n| \geq 0$. Thus, for each positive integer $n \geq 1$, problem (29) has a unique nonnegative solution $v_n \in W_n$ provided that $p_n \geq 2$.

Asymptotic behaviour

Theorem 3.1.2 [D. S.-D., NA-RWA, 2020]

If $\lim_{n\to\infty} p_n = \infty$ and $\limsup_{n\to\infty} \frac{\ln(q_n)}{p_n} < \infty$, the sequence $\{v_n\}$ converges uniformly in Ω to $\mathrm{dist_a}(\cdot, \partial\Omega)$, where $\mathrm{dist_a}(\cdot, \partial\Omega) : \Omega \to [0, \infty)$ is defined by

$$\mathrm{dist_a}(\xi, \partial\Omega) := \inf_{\eta\in\partial\Omega} \left[|x_\xi - x_\eta|_L + |y_\xi - y_\eta|_M\right], \ \forall\, \xi \in \Omega.$$

Step 1: The sequence $\left\{ \int_\Omega v_n(\xi)\, d\xi \right\}$ is bounded.

Step 2: The sequence $\{v_n\}$ is bounded in any $W_0^{1,s}(\Omega)$ with $s > N$.

Step 3: There exists a subsequence of $\{v_n\}$ which converges uniformly in Ω to v_∞.

Theorem [Γ-convergence result]

For each integer $n \geq 1$ consider the functional $H_n : L^1(\Omega) \to [0, \infty]$ defined by

$$H_n(v) = \begin{cases} J_n(v), & \text{if } v \in W_n, \\ +\infty, & \text{otherwise}. \end{cases}$$

We have $\Gamma(L^1(\Omega)) - \lim_{n\to\infty} H_n = H$, where $H : L^1(\Omega) \to [0, \infty]$ is defined by

$$H(v) = \begin{cases} -\int_\Omega f(\xi)v(\xi)\, d\xi, & \text{if } v \in Y, \\ \infty, & \text{otherwise}, \end{cases}$$

with $Y := \Big\{ v \in W^{1,\infty}(\Omega) \cap \Big(\bigcap_{s>1} W_0^{1,s}(\Omega) \Big);$

$$\max\{\| |\nabla_x v(\xi)|_L \|_{L^\infty}, \| |\nabla_y v(\xi)|_M \|_{L^\infty}\} \leq 1 \text{ a.e. } \xi \in \Omega \Big\}.$$

Proposition

Let X be a topological space satisfying the first axiom of countability, and assume that the sequence $\{H_n\}$ of functionals $H_n : X \to \overline{\mathbb{R}}$, Γ-converges to $H : X \to \overline{\mathbb{R}}$. Let z_n be a minimizer for H_n. If $z_n \to z$ in X, then z is a minimizer of H, and

$$H(z) = \liminf_{n\to\infty} H_n(z_n).$$

J. Jost & X. Li-Jost: Calculus of Variations, Cambridge University Press, 2008.

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$$H(z) = \liminf_{n\to\infty} H_n(z_n).$$

We deduce that v_∞ **must be a minimizer for H and in particular,**

$$\max\{\| |\nabla_x v_\infty|_L \|_{L^\infty}, \| |\nabla_y v_\infty|_M \|_{L^\infty}\} \leq 1 \textbf{ a.e. in } \Omega.$$

J. Jost & X. Li-Jost: Calculus of Variations, Cambridge University Press, 2008.

4. Chapter 4: Final comments and further directions of research

An open problem

Let $p, q \in \mathbb{R}$ s.t. $1 < p < q$ and let $0 \le a(\cdot) \in C^{0,\alpha}(\Omega)$, for some $\alpha \in (0,1)$. We define the double phase operator by

$$\Delta_{p,q}^{a(\cdot)} u := \Delta_p u + \mathrm{div}(a(x)|\nabla u|_N^{q-2}\nabla u), \tag{31}$$

that has an ellipticity of order p in the gradient in the points x on the zero set $\{a(x) = 0\}$, while it exhibits a q-growth in the gradient in those points x where $a(x)$ is positive.

We propose to investigate the asymptotic behaviour of the solutions (as $p \to \infty$, and consequently $q \to \infty$) for

$$\begin{cases} -\Delta_{p,q}^{a(\cdot)} u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{32}$$

M. Colombo & G. Mingione (2015): Regularity for double phase variational problems/Bounded minimisers of double phase variational integrals, *Arch. Rational Mech. Anal.* **215 /218**, 443–496/219–273.

Thank you for your attention!!!