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On the semigroup property of the perpendicular bisector in a normed space

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Abstract

Let (X,d) be a metric linear space and *a in X*. Any such point divides the space into three zones: $H_a=\{x \text{ in X: } d(0,x)<d(x,a)\}$, $M_a=\{x \text{ in } X: d(0,x)=d(x,a)\}$, $L_a=\{x \text{ in } X: d(0,x)>d(x,a)\}$. If the distance is generated by a norm, H_a is called the *Leibnizian halfspace* of *a*, M_a is the *perpendicular bisector* of the segment $[0,a]$ and $L_a = a - H_a$. It is known that the perpendicular bisector M_a *is an affine subspace if and only if the norm is generated by an inner product.*

In that case it happens that $(L_a,+)$ is a semigroup. We ask for what type of norms this property still holds. We prove that in two dimensional spaces $(L_a,+)$ is always a semigroup , we prove that in higher dimensions this semigroup property of L_a fails to be true (even for strictly convex norms).

I conjecture that the semigroup property of L_a holds if and only if *X* is an inner product space and wait for an expert to decide if my guess that is true or not.

0. Story of the problem

Let $(X, \|\cdot\|)$ be a real normed space.

In [3] the authors were interested in the following property of *X*, studied in [2]

(A) *For every* $x,y,z \in X$, *the inequalities*

$$\|x\| \geq \|y+z\|,\ \|y\| \geq \|z+x\|,\ \|z\| \geq \|x+y\| \text{ imply the equalities}$$
$$\|x\| = \|y+z\|,\ \|y\| = \|z+x\|,\ \|z\| = \|x+y\|$$

The typical example of spaces with this property are the inner – product spaces. Indeed, if there exists an inner product $<\cdot,\cdot>$ such that $\|x\|^2 = <x,x>$, then the inequalities in (A) become $\|x\|^2 \geq \|y\|^2 + \|z\|^2 + 2<x,y>$, $\|y\|^2 \geq \|x\|^2 + \|z\|^2 + 2<x,z>$, $\|z\|^2 \geq \|y\|^2 + \|x\|^2 + 2<z,y>$; if we add them, we get $\|x+y+z\| \leq 0 \Leftrightarrow x + y + z = 0$. In this case we get even more than the simple equalities $\|x\| = \|y+z\|$, $\|y\| = \|z+x\|$, $\|z\| = \|x+y\|$, namely we have $x = -(y + z)$, $y = -(x + z)$, $z = -(y + x)$.

If we denote $a=x+y+z$ we can write property (A) as

(A) *For every* $x,y,a \in X$, *the inequalities*

$$\|x\| \geq \|a-x\|,\ \|y\| \geq \|a-y\|,\ \|a-x-y\| \geq \|x+y\| \text{ imply the equalities}$$
$$\|x\| = \|a-x\|,\ \|y\| = \|a-y\|,\ \|a-x-y\| = \|x+y\|$$

Written like that, the property **(A)** receives a **geometric flavor**.

Notation. Let $a \in X$. Denote by L_a, M_a, H_a the sets

(1.1) $L_a = \{x \in X \mid \|x-a\| < \|x\|\}$

(1.2) $M_a = \{x \in X \mid \|x-a\| = \|x\|\}$
(1.3) $H_a = \{x \in X \mid \|x-a\| > \|x\|\}$

Notice that M_a is **the perpendicular bisector** of the segment $(0, a)$, L_a is the set of points closer to a and H_a is the set of points from X closer to 0. Some authors [8] call the set H_a **the Leibnizian halfspace of *a*.** Horvath ([8],[9]) writes $H_{0,a}$ instead of H_a , H_a instead of M_a and $H_{a,a}$ instead of L_a.

In the sequel we will assume always that $a \neq 0$ since $a = 0$ is a nonsense.

Using these notations the property **(A)** becomes

(A) *For any* $0 \neq a \in X$, *if* $x \in L_a \cup M_a$, $y \in L_a \cup M_a$,$x+y \in H_a \cup M_a$ *then* $x,y,x+y \in M_a$

A weaker form of this property is

(B) *For any* $0 \neq a \in X$, *if* $x \in L_a \cup M_a$, $y \in L_a \cup M_a$,$x+y \in H_a \cup M_a$ *then* $x+y \in M_a$

But is it possible that $x,y,x+y \in M_a$? In a inner product space that is not possible. In that case $x \in L_a \cup M_a, y \in L_a \cup M_a \Rightarrow x+y \in L_a$.

Or, in other words, $(L_a \cup M_a ,+)$ is a semigroup.

Indeed, squaring the inequalities $\|x\| \geq \|a-x\|$, $\|y\| \geq \|a-y\|$ and adding them we get $0 \geq 2\|a\|^2 - 2<a,x+y>$ hence $\|x+y\|^2 \geq 2\|a\|^2 - 2<a,x+y> + \|x+y\|^2 > \|x+y-a\|^2$, meaning that $x+y$ is in L_a.

1. The main result

A norm is ***strictly convex*** **([1], [7],[8],[9])** if and only if the unity ball $B_1 = \{x \in X : \|x\| \leq 1\}$ is a strict convex set or, equivalently, if the equality $\|x+y\| = \|x\| + \|y\|$ can hold if and only if $x = 0$ or $y = 0$ or $y = \lambda x$ for some $\lambda > 0$.

An equivalent definition is that the norm is a strict convex function i.e. for $x \neq y$, the equality $\|(1-\lambda)x + \lambda y\| = (1 - \lambda)\|x\| + \lambda\|y\|$ can hold if and only if $\lambda \in \{0,1\}$. Here $\lambda \in [0,1]$.

Let us denote by $h_a : X \to \Re$ the mapping $h_a(x) = \|x - a\| - \|x\|$. Then $H_a = \{h > 0\}$, $M_a = \{h = 0\}$ and $L_a = \{h < 0\}$. As the function h_a is continuous, M_a is a closed set and H_a, L_a are open.

Definition. Say that X has the ***semigroup property*** if it satisfies the condition

(M) For any $x,y,a \in X$ the following implication holds

$$\|x\| \geq \|x-a\| , \|y\| \geq \|y-a\| \Rightarrow \|x+y\| \geq \|x+y-a\|$$

Remark. The semigroup property (M) can be restated as

(M) $L_a \cup M_a$ is a semigroup for every $a \in X$

or, equivalently, as $(L_a \cup M_a) + (L_a \cup M_a) \subseteq (L_a \cup M_a)$.

We shall also consider the following property

(M°) L_a *is a semigroup*

Proposition 1.1.

(i). *The property* (M) *implies the property* (B).

(ii) *A space X has the properties* (A), (B), (M) *or* (M°) *if and only if all 3-dimensional subspaces of X have it.*

Proof.

(i). Obvious. If $L_a \cup M_a$ is a semigroup then $x \in L_a \cup M_a, y \in L_a \cup M_a \Rightarrow x+y \in L_a \cup M_a$

Thus, if $x \in L_a \cup M_a$, $y \in L_a \cup M_a$, $x+y \in H_a \cup M_a$ then $x+y \in (H_a \cup M_a) \cap (H_a \cup M_a) = M_a$.

(ii). For instance, if X has the property (A) and Y = span({x,y,z}), then it has obviously the property (A), too. Conversely, if we know that Y has the property (A), it means that X has it, too. □

Remark. Thus the semigroup property is a geometric one, it deals with the shape of the unity ball in a three-dimensional subspace of X. In the case of inner product spaces, this is an ellipsoid. Actually the inner product spaces satisfy a stronger assumption, namely

(C) $\quad \|x\| \geq \|x-a\|,\ \|y\| \geq \|y-a\|,\ \|x+y\| \leq \|x+y-a\| \Rightarrow a = 0$

Of course (C) implies (A).

Any norm space has the following properties:

Proposition 1.2. *Let X be any normed space and* $a \in X$, $a \neq 0$.

(i) *If* $\|x\| \leq \|x-a\|$ *and* $0 \leq t \leq 1$ *then* $\|x\| \leq \|x-ta\|$.

(ii) *If* $x \in L_a$ *and* $t \geq 1$ *then* $tx \in L_a$.

If $x \in M_a$ *and* $t \leq 1$, then $tx \in M_a \cup H_a$.

If $x \in M_a$, $t \geq 1$ *then* $tx \in M_a \cup L_a$.

(iii) *If* $x \in L_a$ *, there exists* $0 \leq t \leq 1$ *such that* $tx \in M_a$

(iv) *If the norm is strictly convex and* $x \in M_a$, *then* $tx \in M_a$ *if and only if* $t = 1$.

Otherwise written, if $x \in M_a$ *and* $t > 1$, *then* $tx \notin M_a$.

(v) *If X is strictly convex, the closures are* $\mathrm{Cl}(H_a) = H_a \cup M_a$ and $\mathrm{Cl}(L_a) = L_a \cup M_a$.

(vi) $L_a = a - H_a$, $H_a = a - L_a$, $M_a = a - M_a$. *The perpendicular bisector is symmetric with respect to a/2 and the symmetric of* L_a *is* H_a.

Proof.

(i). Let f: $\Re \to \Re$ be defined by $f(t) = \|x\| - \|x-ta\|$. The function f is concave , $f(0) = 0$ and $f(1) \geq 0$. Thus $t \in [0,1] \Rightarrow f(t) \geq tf(1) \geq 0 \Rightarrow \|x\| \geq \|x-ta\|$

(ii). Obvious if we write the inequality $\|x\| \geq \|x-ta\|$ as $\|\frac{1}{t}x\| \geq \|\frac{1}{t}x - a\|$. If $t \in (0,1]$, then $\frac{1}{t} \geq 1$. If $x \in M_a$, then $f(0) = f(1) = 0 \Rightarrow f(s) \leq 0\ \forall\ s \in (-\infty,0) \cup (1,\infty) \Leftrightarrow \|x\| \leq \|x - ta\|\ \forall\ t \in (0,1)$. Thus $\|tx\| \leq \|tx-a\|\ \forall\ t \leq 1$.

(iii). The function f from (i) has the property that $f(\infty) = -\infty$.

Indeed, $\|x-ta\| \geq t\|a\| - \|x\| \Rightarrow f(t) \leq 2\|x\| - t\|a\|$. As $f(1) \geq 0$, there must exist $t \geq 1$ such that $f(t) = 0 \Leftrightarrow \|x\| = \|x-ta\| \Leftrightarrow \frac{1}{t}x \in M_a$

(iv). If the norm is strictly convex, then the function f from 1 is strictly concave. Let $s = \frac{1}{t}$.

As $x \in M_a$ is the same as $f(0) = f(1) = 0$, the strict concavity implies $s \in (0,1) \Rightarrow f(s) > 0 \Leftrightarrow \|x\| > \|x - sa\| \Leftrightarrow \|tx\| > \|tx-a\|$.

(v). The inclusion $\mathrm{Cl}(H_a) \subseteq H_a \cup M_a$ holds in any norm space. The problem is to check that $M_a \subseteq \mathrm{Cl}(H_a)$, and that is not true in general. But if X is strictly convex, and $x \in M_a$ then the points $t_n x$ belong to H_a if $t_n > 1$. Now it is obvious : if $t_n \downarrow 1$, then $t_n x \to x$ hence $x \in \mathrm{Cl}(H_a)$. The equality $\mathrm{Cl}(L_a) = L_a \cup M_a$ has the same proof.

(vi). Obvious. $a - x \in H_a \Leftrightarrow \|a-x\| < \|a-(a-x)\| \Leftrightarrow \|a-x\| < \|x\| \Leftrightarrow x \in L_a$ □

The following result also holds in any norm space and it will simplify the things:

Proposition 1.3. *If* $(X, \|\cdot\|)$ *has the property* (M) *and* $T:X \to X$ *is a one to one and onto linear mapping then* $(X, \|\cdot\|_T)$ *has the property* (M), *too, where the norm is defined by*
$\|x\|_T = \|Tx\|$

Moreover, if $(X, \|\cdot\|)$ *is strictly convex, then* $(X, \|\cdot\|_T)$ *is strictly convex, too.*

Proof. Let $H_a^{(T)} = \{ x \in X : \|x\|_T > \|x - a\|_T \}$, $M_a^{(T)} = \{ x \in X : \|x\|_T = \|x - a\|_T\}$ and $L_a^{(T)} = \{ x \in X : \|x\|_T = \|x - a\|_T \}$.

We claim that $H_a^{(T)} = T^{-1}(H_{Ta})$, $M_a^{(T)} = T^{-1}(M_{Ta})$ and $L_a^{(T)} = T^{-1}(L_{Ta})$.

Indeed, $H_a^{(T)} = \{ x \in X : \|Tx\| > \|Tx - Ta\| \} \Leftrightarrow T(H_a^{(T)}) = \{ Tx : \|Tx\| > \|Tx - Ta\| \} = \{y \in X : \|y\| > \|y - a\| \} = H_a$ and the same holds for the other two sets. The property (M) says that $H_a \cup M_a$ is a semigroup for any $a \in X$. Then $H_{Ta} \cup M_{Ta}$ is also a semigroup. But it is obvious that if $H \subseteq X$ is a semigroup and $U:X \to X$ is a linear operator, then $U(H)$ is a semigroup, too. Therefore $H_a^{(T)} \cup M_a^{(T)} = T^{-1}(H_{Ta} \cup M_{Ta})$ is a semigroup .

The last assertion is obvious. □

Example. If $\Re^2$ endowed with some norm has the property (M), the same holds for $\Re^2$ endowed with the norm $\|(x_1, x_2)\|^* = \|(ax_1+bx_2, cx_1 + dx_2)\|$ provided that $ad\text{-}bc \neq$ $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$. Now we can state our main result.

Theorem A. *Let* $X = \Re^2$ *endowed with some norm* $\|\cdot\|$. *Let* $a \in X$. *Then*

(i). $L_a + L_a \subseteq L_a$

(ii). *If X is strictly convex, then* $(L_a \cup M_a) + (L_a \cup M_a) \subseteq L_a \cup M_a$

(iii) *Conversely, if X has the semigroup property, then it is strictly convex.*

Thus in the two-dimensional case the semigroup property is equivalent with the strict convexity.

Proof.

For the sake of a better understanding, it will be divided in several steps.

Step 1. There is no restriction to **consider $a = (0,1)$.**
Indeed, if $a = (a_1, a_2)$, we can choose a linear operator T such that $Ta = (0,1)$. After that replace the norm $\|\cdot\|$ with the norm $\|\cdot\|_T$ and use Proposition 1.3.

Step 2. The two-dimensional norms have the **following useful property** (which maybe is well known, but we prove it here since we didn't find appropriate references and it can be used to construct many norms on the plane):

Lemma 1.4. ***Let*** $\|.\|$ *be a norm on* $X = \Re^2$ *and* $f: \Re \to (0,\infty)$ *defined by* $f(t) = \|(1,t)\|$
Then f is convex and

(1.4) $$\lim_{t\to\infty} \frac{f(t)}{t} = \lim_{t\to-\infty} \frac{f(t)}{-t} = m > 0$$

(1.5) $$\|(x,y)\| = \begin{cases} |x|\, f\left(\dfrac{y}{x}\right) & if \quad x \neq 0 \\ m\,|y| & f \quad x = 0 \end{cases}$$

Moreover, X is strictly convex, if and only if the function f is strictly convex.

Conversely, for any convex function $f\colon \Re \to (0,\infty)$ *satisfying* (1.4), *the equality* (1.5) *defines a norm.*

All the norms on a two dimensional space have this form.

If f is strictly convex, then the norm given by (1.5) *is strictly convex, too.*

For the proof of the Lemma see Appendix.

Combine Step 1 and Step 2. Thus $a = (0,1)$ and f is a function as in the above Lemma. Then we drop the index a and write

$L = \{(x,y) : x \neq 0,\ f\left(\frac{y}{x}\right) > f\left(\frac{y-1}{x}\right)$ or $x = 0,\ m|y| > m|y-1| \Leftrightarrow y \in (½ , \infty)\}$,

$M = \{(x,y) : x \neq 0 ,\ f\left(\frac{y}{x}\right) = f\left(\frac{y-1}{x}\right)$ or $x = 0,\ m|y| = m|y-1| \Leftrightarrow y = ½ \}$

$H = \{(x,y) : x \neq 0,\ f\left(\frac{y}{x}\right) < f\left(\frac{y-1}{x}\right)$ or $x = 0,\ m|y| < m|y-1| \Leftrightarrow y \in (-\infty, ½)\}$

Step 3. Describe the sets *M, H, L*.

Due to the property that $\lim_{t\to\infty}\frac{f(t)}{t} = \lim_{t\to-\infty}\frac{f(t)}{-t} = m > 0$, the convex function f behaves as follows: there exists $u_1 \le u_2 \in \Re$ such that

(1.6) f is decreasing on $(-\infty, u_1)$

(1.7) f *is* constant on $[u_1,u_2]$

(1.8) f *is* increasing on $[u_2 , \infty)$

Note that if f is strictly convex, then $u_1 = u_2$. Let $v = f(u_1) = f(u_2)$.

Lemma 1.5. *Let* $f : \Re \to \Re$ *be a continuous function which satisfies the assumptions* (1.6), (1.7), (1.8). *Suppose further that* $\lim_{t\to\infty}\frac{f(t)}{t} = m_2 > 0,\ \lim_{t\to-\infty}\frac{f(t)}{-t} = m_1 > 0$

Let $h(x,y) = f\left(\frac{y}{x}\right) - f\left(\frac{y-1}{x}\right)$, $f : \Re^*\times\Re \to \Re$.

Then $M = \{h = 0\}$, $L = \{h > 0\}$, $H = \{h < 0\}$ *may be characterized by:*

There exist two functions $\varphi_1, \varphi_2\colon \Re^* \to \Re$ *, such that*

(1.9) $L = \{(x,y) : \varphi_1(x)\vee\varphi_2(x) < y\}$

(1.10) $M = \{(x,y)\colon \varphi_1(x)\wedge\varphi_2(x) \le y \le \varphi_1(x)\vee\varphi_2(x) \}$

(1.11) $H = \{(x,y) : \varphi_1(x)\wedge\varphi_1(x) > y\}$

Moreover,

(1.12) $\varphi_j(-x) = 1 - \varphi_j(x)\ \forall\ x \in \Re^*$, $j = 1,2$

(1.13) $\varphi_j(0+0)= \frac{m_1}{m_1+m_2}$,$\varphi_j(0-0)= \frac{m_2}{m_1+m_2}$

(1.14) $\varphi_1(x) = \varphi_2(x)\ \forall\ x \in (-\frac{1}{u_2-u_1},\frac{1}{u_2-u_1})\setminus\{0\}$

(1.15) *If f is strictly convex, then* $\varphi_1 = \varphi_2$

Finally,

(1.16) *if* $(x,y) \in M$ *and* $(tx,ty) \in M$ for some $t > 1$, then either $x > \frac{1}{u_2 - u_1}$ or $x < -\frac{1}{u_2 - u_1}$.

For the proof see the appendix.

In our case $m_1 = m_2$ allows us to extend the mapping φ_j to 0, defining $\varphi_j(0) = ½$.
Recall our claim: we want to prove that

(1.17) $y > \varphi_1(x) \vee \varphi_2(x)$, $y' > \varphi_1(x') \vee \varphi_2(x') \Rightarrow y + y' > \varphi_1(x+x') \vee \varphi_2(x+x')$

This implication would surely hold if we could prove that φ_j are sub-additive. Indeed, in that case $y > \varphi_1(x)$, $y' > \varphi_2(x') \Rightarrow y + y' > \varphi_1(x) + \varphi_1(x') \geq \varphi_1(x+x')$ and similarly $y+y' > \varphi_2(x+x')$.

Step 4. The functions φ_j are sub-additive

We shall use the following criterion for sub-additivity:

Lemma 1.6.

(i). *Let* $g:\Re \to \Re$ *be such that* $g(\lambda s) \leq \lambda g(s)$ $\forall \lambda \geq 1$ $\forall s \in \Re$. *Then* $g(s+t) \leq g(s) + g(t)$ $\forall s,t$ *such that* $st \geq 0$. *Moreover, if* $\lambda > 1 \Rightarrow g(\lambda s) < \lambda g(s)$ $\forall s \in \Re$, *then* $g(s+t) < g(s) + g(t)$ $\forall s,t$ *such that* $st \geq 0$.

(ii). *Let g be a function as before. Suppose, that g satisfies the following symmetry property*:

(S) *there exists* $a \in \Re$, $b > 0$ *such that* $g(a - x) + g(x) = b$.

Then g is subadditive: $g(x + y) \leq g(x) + g(y)$.

Moreover, if g satisfies the stronger condition $\lambda > 1 \Rightarrow g(\lambda s) < \lambda g(s)$ $\forall s \in \Re$, *then g is strictly subadditive:* $g(s+t) < g(s) + g(t)$ $\forall s,t$

For the proof, see the Appendix.

Proposition 1.7. *The functions* φ_j *defined by* Lemma (1.5) *are sub-additive.*
Moreover, *if X is strictly convex, they coincide and are strictly sub-additive.*
Proof.

Because of the fact that $m_1 = m_2$, these two functions are defined on the whole real line ($\varphi_j(0) = ½$) and they coincide on the interval $[-\frac{1}{u_2 - u_1}, \frac{1}{u_2 - u_1}]$. The equality (1.12) points out that the symmetry property (S) is fulfilled with $a = 0$, $b = 1$. Thus we only have to prove that

(1.18) $\varphi_j(\lambda x) \leq \lambda \varphi_j(x)$ for any x and $\lambda > 1$.

Check that for φ_1. Indeed,

- if $x < -\frac{1}{u_2 - u_1}$, then $\varphi_1(x) = 1 + xu_2$, $\varphi_1(\lambda x) = 1 + \lambda x u_2 < \lambda + \lambda x u_2 = \lambda \varphi_1(x)$;
- if $x > \frac{1}{u_2 - u_1}$, $\varphi_1(x) = xu_2 \Rightarrow \varphi_1(\lambda x) = \lambda \varphi_1(x)$.
- if $x \in [-\frac{1}{u_2 - u_1}, \frac{1}{u_2 - u_1}]$, then $y = \varphi_1(x) \Leftrightarrow (x,y) \in M$. Then $(\lambda x, \lambda y) \in L \cup M$ due to proposition 1.2(ii). But (tx,ty) does **not** belong to M because of (1.16) hence in this case $(tx,ty) \in L$. According to (1.9) this means that $\lambda y > (\varphi_1 \vee \varphi_2)(\lambda x) \Rightarrow \lambda \varphi_1(x) > \varphi_1(\lambda x)$.

In the same way one proves that $\varphi_2(\lambda x) \leq \lambda \varphi_2(x)$ $\forall \lambda \geq 1$.

If X is strictly convex, then if $(x,y) \in M$ and $\lambda > 1$, then $(\lambda x, \lambda y) \in L \Leftrightarrow \varphi(\lambda x) < \lambda y = \lambda \varphi(x)$ hence Lemma 1.6 points out that φ is strictly sub-additive.

This ends the proof of the proposition.□

Now the proof of Theorem A immediately follows due to Lemma 1.4(iii).

Step 5. Theorem A: end of proof

(i). According to (1.9), $(x,y) \in L \Leftrightarrow y > \varphi(x)$ with $\varphi = \varphi_1 \vee \varphi_2$. The functions φ_j are sub-additive hence φ is sub-additive, too. Therefore $y > \varphi(x)$, $y' > \varphi(x') \Rightarrow y + y' > \varphi(x) + \varphi(x') \geq \varphi(x+x') \Rightarrow (x+x', y+y') \in L$.

(ii). If the space X is strictly convex, $\mathrm{Cl}(H) = H \cup M$. But obviously

$H + H \subseteq H \Rightarrow \mathrm{Cl}(H) + \mathrm{Cl}(H) \subseteq \mathrm{Cl}(H)$.

This ends the proof.

(iii). Suppose that X is **not** strictly convex. Then the unity sphere $B = \{z \in X : \| z \| =1\}$ contains a segment of line $I = \{(1-\lambda)z_1 + \lambda z_2 : 0 \leq \lambda \leq 1\}$ for some $z_1 \neq z_2 \in B$. We suppose these two points to be extreme ones. Let $T:\Re^2 \to \Re^2$ a linear operator such that $Tz_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $Tz_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Suppose that $(X, \| \cdot \|)$ has the semigroup property. Then $(X, \| \cdot \|_T)$ has it, too. The new unity sphere $B_T = \{z \in X : \| z \|_T =1\}$contains the segment $\{(1,t) \mid -1 \leq t \leq 1\}$.

Let $f(t) = \left\| \begin{pmatrix} 1 \\ t \end{pmatrix} \right\|_T$. This function is convex and decreasing on $(-\infty, -1]$, constant on $[-1,1]$ and increasing on $[1,\infty)$. Thus u_1 and u_2 from Lemma 1.5 are $u_1 = -1$, $u_2 = 1$. According to (1.10), $z = (x,y)$ belongs to M if and only if

$(\varphi_1 \wedge \varphi_2)(x) \leq y \leq (\varphi_1 \vee \varphi_2)(x)$. Moreover, due to (3.17), $x \in [\,½ ,\infty) \Rightarrow \varphi_1(x) = x$, $\varphi_2(x) = 1 - x$ and $x \in (-\infty, ½\,] \Rightarrow \varphi_1(x) = 1+ x$, $\varphi_2(x) = - x$. It follows that there are many pairs z,z' such that $z,z' \in M$ but $z+z' = 0 \in L$ (for instance $z = (1,0)$ and $z' = (-1,0)$ are both in M!). This contradicts the semigroup property: $M + M$ should be included in $L \cup M$.

To conclude: if $a = T^{-1}\mathbf{e}_2$, then one can find $z \in M_a$ such that $-z \in M_a$, too. □

We can even prove more.

Theorem B. *The space* $X = \Re^2$ *endowed with a strict convex norm satisfies the assumption* (C)

Proof. We have to prove that the relations

(1.19) $\| z \| \geq \| z - a \|$, $\| z' \| \geq \| z' - a \|$, $\| z + z' \| = \| z + z' - a \|$

are possible for some $z, z' \in X$ if and only if $a = 0$. Suppose, ad absurdum, that this is no true. Let $a = (a_1, a_2) \neq 0$. There exists a one-one and onto linear operator $T : X \to X$ such that $Ta = \mathbf{e}_2 = (0,1)$. Let $w = T^{-1}z$, $w' = T^{-1}z'$. Then (1.19) becomes

(1.20) $\| w \|_T \geq \| w - \mathbf{e}_2 \|_T$, $\| w \|_T \geq \| w' - \mathbf{e}_2 \|_T$, $\| w + w' \|_T = \| w + w' - \mathbf{e}_2 \|_T$

where $\| \cdot \|$ is the norm defined at Proposition 1.3.

If X is strictly convex, the norm $\| \cdot \|_T$ is strictly convex, too. Let $w = (x,y)$, $w' = (x',y')$ and φ the function defined by Lemma 1.5. Then (1.20) become

(1.21) $y \geq \varphi(x)$, $y' \geq \varphi(x')$, $y + y' = \varphi(x + x')$

But this is impossible since $y + y' \geq \varphi(x) + \varphi(x') > \varphi(x + x')$, as φ is strictly sub-additive.□

2. Conjectures, open problems and counterexamples

At a first glance, a ***possible conjecture*** would be that a strict convex normed space should have the semigroup property. But this not true: all the spaces $L^p(\{1,2,3\})$ with $1<p<\infty$ are strict convex and fail to have the property (M).

Proposition 2.1 *Let* $X = \Re^3$ endowed with l^p norm, $\|x\| = \left(\sum_{j=1}^{3} |x_j|^p\right)^{\frac{1}{p}}$.

(i). *If* $p \in [1,2) \cup (2,\infty]$ *, then X does* ***not*** *have the semigroup property.*
(ii). *However, all its proper subspaces* ***have it*** (*due to Theorem A*).
Proof.

(i)..Case 1: $p \in [1,2)$. We choose $x = (3,t,-t)$, $y = (3, -t, t)$, $a = (4,4,4)$. Then $x + y = (6, 0, 0)$, $x - a = (-1, t - 4, - t - 4)$, $y - a = (-1, -t - 4, t -4)$, $x + y - a = (2, - 4, - 4)$ hence, if $t > 4$, $\|x\|^p = \|y\|^p = 3^p + 2t^p$, $\|x - a\|^p = \|y - a\|^p = 1 + (t - 4)^p + (t + 4)^p$, $\|x+y\|^p = 6^p$ and $\|x+y - a\|^p = 2^p + 2\cdot 4^p$. We claim that for any $1 \le p < 2$ we can choose $t > 4$ such that $\|x\| > \|x - a\|$ and $\|y\| > \|y - a\|$ but $\|x + y\| < \|x + y - a\|$. Indeed, this is equivalent to
$3^p - 1 > (t - 4)^p + (t + 4)^p - 2t^p$ but $6^p < 2^p + 2\cdot 4^p$. The function $g(t) = (t - 4)^p + (t + 4)^p - 2t^p$ has the property that $g(\infty) = 0$ for any $1 \le p < 2$, hence for any fixed $p \in [1,2)$ we can find a $t = t(p)$ such that $3^p - 1 > g(t)$. As about the second condition, $6^p < 2^p + 2\cdot 4^p \Leftrightarrow 3^p < 1 + 2\cdot 2^p$, it holds for any such $p \in [1,2)$ since the function $h(p) = 1 + 2\cdot 2^p - 3^p$ is decreasing on $[1,2)$ and positive.

Case 2: $p \in (2,\infty]$. Now we choose $x = (1,-1,2)$, $y = (1,2,-1)$, $a = (-t,t,t)$ for some $t \in (0, \frac{1}{2})$. Then $x + y = (2,1,1)$, $x - a = (1+t,-1-t,2-t)$, $y - a = (1+t,2-t,-1-t)$, $x + y - a = (2+t,1-t, 1-t)$ hence $\|x\|^p = \|y\|^p = \|x + y\|^p = (2^p+2)$, $\|x - a\|^p = \|y - a\|^p = 2(1+t)^p + (2 - t)^p$ and $\|x+y - a\|^p = (2 + t)^p + 2(1 - t)^p$. We claim that for every $p > 2$ there exists $t \in (0, \frac{1}{2})$ such that $\|x\| > \|x-a\|$, $\|y\| > \|y-a\|$ but $\|x + y\| < \|x + y - a\|$. Indeed, let $p > 2$ be fixed and let f,g be defined by $f(t) = 2^p + 2 - 2(1+t)^p - (2 - t)^p$, $g(t) = (2 + t)^p + 2(1 - t)^p - 2^p - 2$. Notice that $f(0) = g(0) = 0$ and that $f'(0) = g'(0) = p(2^{p-1} - 2)$. If $p > 2$, then $2^{p-1} > 2$, hence the derivatives are positive. It means that for small t we have $f(t) > 0$, $g(t) > 0$; this fact agree to our claim. For $p = \infty$, it is even simpler, since now $\|x\| = \|y\| = \|x + y\| = 2$, $\|x - a\| = \|y - a\| = 2 - t$ and $\|x + y - t\| = 2 + t$.

Thus the two-dimensional space, $\Re^2$ endowed with the norm l^p, $1 < p < \infty$ has the semigroup property while $\Re^3$ endowed with the same norm **doesn't have it**.

Open problems.

1. The only three dimensional spaces which satisfy the semigroup property (B) are inner product spaces. **Prove or disprove that if dim(*X*) ≥ 3, then property (M) implies the fact that *X* has an inner product**. In other problems connected with perpendicular bisectors that was indeed the case (see [1],[7],[8],[9]). For instance one knows that if all the perpendicular bisectors are semispaces, then *X* is ab inner product space([10], theorem 5.4) . It is also known that if all the Leibnizian halfspaces H_a are convex, then *X* is again inner product space [8],[9],[10].
2. The only examples of spaces having the semigroup property also satisfy the property *C*. Prove or disprove that (M) ⇔ (C)
3. Prove or disprove that $M + M \subseteq M \cup L \Rightarrow X$ has the property (M)

Appendix.

***Proof of Lemma* 1.4.**

Let $f(t)$ = ‖ $(1,t)$ ‖ . Obviously f is convex, $\lim_{t\to\infty}\frac{f(t)}{t}=\lim_{t\to-\infty}\frac{f(t)}{-t}=\|(0,1)\|:=m>0$ and if $x\neq 0$ we can write ‖ (x,y) ‖ $=|x|\left\|\left(1,\frac{y}{x}\right)\right\|$. If $x = 0$ then ‖ $(0,y)$ ‖ = $|y|$ ‖ $(0,1)$ ‖ = $m|y|$.

Conversely, let $f:\Re\to(0,\infty)$ be a convex function such that $\lim_{t\to\infty}\frac{f(t)}{t}=\lim_{t\to-\infty}\frac{f(t)}{-t}=m>0$. We claim that the equality

(3.1) $$p(x,y)=\begin{cases}|x|\,f\left(\frac{y}{x}\right) & if \quad x\neq 0\\ m\,|y| & f \quad x=0\end{cases}$$

defines a norm: it is sub-additive, $p(x,y) = 0 \Leftrightarrow x = y = 0$ and $p(tx,ty) = |t|p(x,y)$. As the last two assertions are obvious, we shall focus on the sub-additivity. One must prove that

(3.2) $$|x+x'|\,f\left(\frac{y+y'}{x+x'}\right)\leq |x|\,f\left(\frac{y}{x}\right)+|x'|\,f\left(\frac{y'}{x'}\right) \text{ if } x,x' \text{ and } x+x'\neq 0$$

(3.3) $$|x|\,f\left(\frac{y+y'}{x}\right)\leq |x|\,f\left(\frac{y}{x}\right)+m|y'| \text{ if } x\neq 0,\ x'=0$$

(3.4) $$|x'|\,f\left(\frac{y+y'}{x'}\right)\leq |x'|\,f\left(\frac{y'}{x'}\right)+m|y| \text{ if } x'\neq 0,\ x=0$$

(3.5) $$m|y+y'|\leq |x|\left(f\left(\frac{y}{x}\right)+f\left(\frac{-y'}{x}\right)\right) \text{ if } x+x'=0,\ x\neq 0$$

Of course, the most important is (3.2). If both x and x' are positive, it is easy. Indeed, $p(x+x',y+y') = (x+x')f\left(\frac{y+y'}{x+x'}\right)=(x+x')f\left(\frac{x}{x+x'}\cdot\frac{y}{x}+\frac{x'}{x+x'}\cdot\frac{y'}{x'}\right)\leq |x|\,f\left(\frac{y}{x}\right)+|x'|\,f\left(\frac{y'}{x'}\right)$ due to convexity. But if one of them is negative, this argument doesn't work anymore. We shall use another one.

We shall prove that any convex function $f:\Re\to(0,\infty)$ such that $\lim_{t\to\infty}\frac{f(t)}{t}=\lim_{t\to-\infty}\frac{f(t)}{-t}=m>0$ admits the following representation

(3.6) $$f(t) = a+\int|t-c|\,d\nu(c)$$

where ν is a measure on the real line such that $\nu(\Re) = m$ and $\mu_1 := \int|c|\,d\nu(c)<\infty$

It is well known (see for instance [4] pg 96, formula (3.3)) that any convex function $g:[0,\infty)\to\Re$ which is differentiable at 0 admits a representation of the form

(3.7) $$g(t) = a+bt+\int(t-c)_+\,d\nu(c)$$

where ν is a Stieltjes measure concentrated on $(0,\infty)$. If $b\geq 0$ we can replace ν with $\nu + b\delta_0$ (here δ_0 is the Dirac measure concentrated at 0) to obtain a simpler relation

(3.8) $$g(t) = a+\int(t-c)_+\,d\nu(c),\ \mathrm{Supp}(\nu)\subseteq[0,\infty)$$

Now let us look at our function $f:\Re\to(0,\infty)$. It is convex and non-increasing on some interval $(-\infty, u]$, non-decreasing on $[u,\infty)$. Suppose that $u = 0$. Then there exist two Stieltjes measures ν_1 and ν_2, $\mathrm{Supp}(\nu_1)\subseteq(-\infty, 0]$, $\mathrm{Supp}(\nu_2)\subseteq[0,\infty)$ such that

(3.9) $$f(t)=\begin{cases}a+\int(-t+c)_+\,d\nu_1(c) & if \quad t<0\\ a+\int(t-c)_+\,d\nu_2(c) & if \quad t\geq 0\end{cases}$$

As $\lim_{t\to\infty}\frac{f(t)}{t}=\lim_{t\to-\infty}\int\left(1-\frac{c}{t}\right)_{+}d\nu_2(t)=\nu_2([0,\infty))=\nu_2(\Re)$ (we applied Beppo- Levi's theorem) and

$\lim_{t\to-\infty}\frac{f(t)}{-t}=\lim_{t\to\infty}\int\left(1+\frac{c}{t}\right)_{+}d\nu_1(t)=\nu_1((-\infty,0])=\nu_1(\Re)$, we get $\nu_1(\Re)=\nu_2(\Re):=m$

Now use the relation $2x_+ = |x| + x$. Then (3.9) becomes

(3.10) $$2f(t)-2a=\begin{cases}\int|t-c|d\nu_1(c)-tm+\int c\,d\nu_1(c) & if \quad t<0\\ \int|t-c|\,d\nu_2(c)+tm-\int c\,d\nu_2(c) & if \quad t\geq 0\end{cases}$$

Write (3.10) as

(3.11) $$2f(t)-2a=\begin{cases}\int|t-c|d\nu_1(c)+|t|m-\int|c|\,d\nu_1(c) & if \quad t<0\\ \int|t-c|\,d\nu_2(c)+|t|m-\int|c|\,d\nu_2(c) & if \quad t\geq 0\end{cases}=\begin{cases}f_1(t) & if \quad t<0\\ f_2(t) & if \quad t\geq 0\end{cases}$$

Now remark that $t \geq 0 \Rightarrow f_1(t) = 0$ and $t \leq 0 \Rightarrow f_2(t) = 0$ (this should be obvious if we look at (3.9)!). It means that we can write $2f(t) - 2a = f_1 + f_2 = \int|t-c|d\nu(c)+2m|t|-\int|c|\,d\nu(c)$.

Therefore we arrived at a representation of the form

(3.12) $f(t) = a + m|t| + \int|t-c|\,d\mu(c)$

which holds for any convex function non-increasing on $(-\infty,0)$, non-decreasing on $(0,\infty)$ such that $\lim_{t\to\infty}\frac{f(t)}{t}=\lim_{t\to-\infty}\frac{f(t)}{-t}=m>0$. If we replace the measure $\mu = \nu/2$ by $\mu + m\delta_0$ we arrive at the formula (3.6). After that we can replace our assumption that f is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$ by "non-increasing on $(-\infty,u)$, non-decreasing on (u,∞)" for some $u \in \Re$ and the proof remains the same.

Therefore any convex function as in Lemma 1.4 can be represented by some finite Stieltjes measure by formula (3.6).

Let $\mathcal{C} = \{f : \Re \to [0,\infty) : f \text{ satisfies conditions } (3.2) - (3.5)\}$

Then $\mathcal{C}$ contains all the positive constants (for them $m = 0$!) and all the functions $f(x) = |x - a|$ (indeed now $m = 1$: $|x+x'|f\left(\frac{y+y'}{x+x'}\right) = |(y - ax) +(y' - ax')|$, $|x|\,f\left(\frac{y}{x}\right) = |y - ax|$, $|x'|\,f\left(\frac{y'}{x'}\right) = |y' - ax'|$, the inequality $|x|f\left(\frac{y+y'}{x}\right)\leq |x|\,f\left(\frac{y}{x}\right)+m|y'|$ becomes $|y+y'-ax| \leq |y - ax| + |y'|$ a.s.o)

and is a ***cone*****:** $f,g \in \mathcal{C} \Rightarrow af + bg \in \mathcal{C}\ \forall\ a,b \geq 0$. Moreover, it is *closed* with respect to pointwise convergence. As any integral is a limit of finite sums, $\mathcal{C}$ contains all the functions of the form (3.6). This ends the proof. □

***Proof of Lemma* 1.5.** Let $f(\Re)=[v,\infty)$, and $f^{-1}(\{v\}) = [u_1, u_2]$.

Let $f_1 : (-\infty, u_1] \to [v, \infty)$ and $f_2 : [u_2 , \infty) \to [v,\infty)$ be defined by $f_j(x) = f(x)$. These functions are invertible. We know that $M = \{(x,y) \mid f\left(\frac{y}{x}\right)=f\left(\frac{y-1}{x}\right)\}$. There are two possibilities

- Case 1. $x > 0$. Now $\frac{y-1}{x}<\frac{y}{x}$ and the equality $f\left(\frac{y}{x}\right)=f\left(\frac{y-1}{x}\right)$ is possible only if $u_1 \leq \frac{y-1}{x}<\frac{y}{x}\leq u_2$ or if $\frac{y-1}{x}<u_1<\frac{y}{x}<u_2$.

In the first case $x \geq \frac{1}{u_2-u_1}$ and $1 + xu_1 \leq y \leq xu_2$.

In the second one, $f\left(\frac{y}{x}\right) = f_2\left(\frac{y}{x}\right)$ and $f\left(\frac{y-1}{x}\right) = f_1\left(\frac{y-1}{x}\right)$ hence $(x,y) \in M \Rightarrow$ $f_1\left(\frac{y-1}{x}\right) = f_2\left(\frac{y}{x}\right)$. Let $t > v$, $t = f\left(\frac{y}{x}\right)$. Therefore $y = xf_2^{-1}(t)$, $y - 1 = xf_1^{-1}(t)$, thus the set M can be described by the parametric curve

(3.13) $$x(t) = \frac{1}{f_2^{-1}(t) - f_1^{-1}(t)}, y(t) = \frac{f_2^{-1}(t)}{f_2^{-1}(t) - f_1^{-1}(t)}, v < t < \infty .$$

This is the graph of some function $\psi_2:(0, \frac{1}{u_2 - u_1}] \to \Re$ with

(3.14) $$\psi_2\left(\frac{1}{u_2 - u_1}\right) = \frac{u_2}{u_2 - u_1}$$ and

$$\varphi(0+0) = \frac{m_1}{m_1 + m_2} .$$

Indeed, $\lim_{t\uparrow\infty} \frac{f_2^{-1}(t)}{f_2^{-1}(t) - f_1^{-1}(t)} = \lim_{t\uparrow\infty} \frac{f_2^{-1}(t)/t}{f_2^{-1}(t)/t - f_1^{-1}(t)/t} = \lim_{t\uparrow\infty} \frac{f_2^{-1}(t)/t}{f_2^{-1}(t)/t - f_1^{-1}(t)/t}$

As $\lim_{t\uparrow\infty} \frac{f_2^{-1}(t)}{t} = \lim_{x\uparrow\infty} \frac{x}{f_2(x)} = \frac{1}{m_2}$ and $\lim_{t\uparrow\infty} \frac{f_1^{-1}(t)}{t} = \lim_{x\uparrow\infty} \frac{-x}{f_1(x)} = -\frac{1}{m_1}$, we get

$$\varphi(0+0) = \frac{1/m_2}{1/m_2 + 1/m_1} = \frac{m_1}{m_1 + m_2} .$$

- Case 2. $x < 0$. Now $\frac{y-1}{x} > \frac{y}{x}$ and the equality $f\left(\frac{y}{x}\right) = f\left(\frac{y-1}{x}\right)$ is possible only if $u_1 \le \frac{y}{x} < \frac{y-1}{x} \le u_2$ or if $\frac{y}{x} < u_1 < \frac{y-1}{x} < u_2$.

In the first case $x \le -\frac{1}{u_2 - u_1}$ and $1 + xu_2 \le y \le xu_1$.

In the second one, $(x,y) \in M \Rightarrow f_1\left(\frac{y}{x}\right) = f_2\left(\frac{y-1}{x}\right) = t$ for some $t > v$. Thus $y = 1 + xf_2^{-1}(t) = xf_1^{-1}(t)$ and we have the parametric description

(3.15) $$x^*(t) = -\frac{1}{f_2^{-1}(t) - f_1^{-1}(t)}, y(t) = 1 - \frac{f_2^{-1}(t)}{f_2^{-1}(t) - f_1^{-1}(t)}, v < t < \infty .$$

Comparing this to (3.13), we see that the curve $(x^*(t),(y^*(t))_{t>v}$ is the graph of some function $\psi_1:[-\frac{1}{u_2 - u_1}, 0)$ which is the symmetric of the graph of ψ_2 with respect to (0, ½):

(3.16) $$\psi_1(-x) = 1 - \psi_2(x) \ \forall\ x \in (0, \frac{1}{u_2 - u_1}].$$

Add all this facts together. Let $\varphi_j : \Re^* \to \Re$ be defined by

$$(3.17)\quad \varphi_1(x) = \begin{cases} 1+xu_2 & if \quad x < -\frac{1}{u_2-u_1} \\ 1-\psi_2(-x) & -\frac{1}{u_2-u_1} \le x < 0 \\ \psi_2(x) & if \quad 0 < x \le \frac{1}{u_2-u_1} \\ xu_2 & if \quad x > \frac{1}{u_2-u_1} \end{cases}, \quad \varphi_2(x) = \begin{cases} xu_1 & if \quad x < -\frac{1}{u_2-u_1} \\ 1-\psi_2(-x) & -\frac{1}{u_2-u_1} \le x < 0 \\ \psi_2(x) & if \quad 0 < x \le \frac{1}{u_2-u_1} \\ 1+xu_1 & if \quad x > \frac{1}{u_2-u_1} \end{cases}.$$

Then $M = \{h = 0\} = \{(x,y) \in \Re^*\times\Re : \varphi_1(x) \wedge \varphi_2(x) \le y \le \varphi_1(x) \vee \varphi_2(x)\}$

Notice that if $f\, u_1 = u_2$, then $\varphi_1 = \varphi_2$. This happens, for instance, if f is strictly convex.

Now let us determine the set $L = \{h > 0\} = \{(x,y) : f\left(\frac{y}{x}\right) > f\left(\frac{y-1}{x}\right)\}$. If $x > 0$, this set surely contains all the points from $\{(x,y) \in (0,\infty)\times\Re\colon \frac{y}{x} > \frac{y-1}{x} > u_2\}$ since on the interval $[u_2,\infty)$ the function f is increasing. Thus L includes the set $\{(x,y) \in (0,\infty)\times\Re\colon y > 1 + xu_2\}$. If $x < 0$, it contains the set $\{(x,y) \in (-\infty,0)\times\Re\colon \frac{y}{x} < \frac{y-1}{x} < u_1\}$ since on $(-\infty,u_1]$ the function f is decreasing. To conclude, $L \supseteq \{(x,y) \in \Re^*\times\Re\colon y > \max(1+ xu_1, 1 + xu_2)\}$.

Similarly, $H \supseteq \{(x,y) \in \Re^*\times\Re\colon y < \min(xu_1, xu_2)\}$.

We found some points from L are above the graph of $\varphi_1\vee\varphi_2$ and some points from H are below it. It means that ***all*** the points from L are in the set $\{(x,y)\in \Re^*\times\Re : y > \varphi_1(x) \vee \varphi_2(x)\}$ and ***all*** the points from L are in the set $\{(x,y) \in \Re^*\times\Re : y > \varphi_1(x) \vee \varphi_2(x)\}$. Because, suppose ad absurdum that L contains a point (x_0,y_0) such that $y_0 < (\varphi_1\wedge\varphi_2)(x_0)$. There exist some $c > 0$ such that $y_0 - c < \min(xu_1, xu_2) \Rightarrow (x_0, y_0 - c) \in H$. By Darboux theorem, on the segment joining the points (x_0, y_0) and $(x_0, y_0 - c)$ there must exist at least one, let us say (x_0, η) such that $h(x_0, \eta) = 0$ $\Leftrightarrow (x_0, \eta) \in M$, contradicting he fact that $(x,\eta) \in M \Leftrightarrow \varphi_1(x_0) \wedge \varphi_2(x_0) \le \eta \le \varphi_1(x_0) \vee \varphi_2(x_0)$.

To conclude, $L = \{(x,y) : y > (\varphi_1\vee\varphi_2)(x)\}$ and $H = \{(x,y) : y < (\varphi_1\wedge\varphi_2)(x)\}$.

The situation simplifies a lot if $u_1 = u_2$ (for instance if f is strictly convex): in that case $\varphi_1=\varphi_2=\varphi$, M is the graph of φ, L are the points above the graph and H are the points below it.

Now prove the claim (3.16). Suppose that $(x,y) \in M$, $(tx, ty) \in M$ for some $t > 1$. Thus $f\left(\frac{y}{x}\right) = f\left(\frac{y-1}{x}\right) = f\left(\frac{ty-1}{tx}\right)$. But the only possibility that the equation $f(x) = \alpha$ have more than two solutions is that $t = v$ hence these solutions be included in $[u_1, u_2]$. It follows either that $x > 0$ $\frac{y-1}{x} < \frac{ty-1}{tx} < \frac{y}{x} \Leftrightarrow x \ge \frac{1}{u_2-u_1}$, or that $x < 0$ and $\frac{y-1}{x} > \frac{ty-1}{tx} > \frac{y}{x} \Leftrightarrow x \le -\frac{1}{u_2-u_1}$.

□

Proof of Lemma 1.6.

It uses the following inequality, which maybe is interesting for itself

Lemma 3.1. (*An elementary inequality*)

Let $a,b \in\Re$. Then $\min[a(1 + \frac{s}{t}), b(1 + \frac{t}{s})] \le a + b$ *for any reals s,t such that* $st > 0$.

Proof. Let $\lambda = \frac{s}{t} > 0$. We want to check that $a(1+\lambda) \le a + b$ or $b(1 + \frac{1}{\lambda}) \le a + b \Leftrightarrow a\lambda \le b$ or $\frac{b}{\lambda} \le a \Leftrightarrow a\lambda \le b$ or $b \le a\lambda$. Obvious.\

The proof of Lemma 1.6 goes as follows:

(i). Notice that $g(0) = g(\lambda\cdot 0) \le \lambda g(0)\ \forall\ \lambda \ge 1$ hence $g(0) \ge 0$ and if g satisfies the stronger condition $\lambda > 1 \Rightarrow g(\lambda x) > \lambda g(x)$, then $g(0) > 0$.

Let $s,t \in \Re$ such that $st \ge 0$. If $s = 0$ the claim becomes $g(t) \le g(t) + g(0) \Leftrightarrow g(0) \ge 0$, true. If g satisfies the stronger condition $g(\lambda s) < \lambda g(s)\ \ \forall\ \lambda > 1$, then this implies $g(0) > 0$, true. Therefore if $s = 0$ we have proved that $g(s+t) \le g(s) + g(t)$. If $t = 0$ the proof is the same. Suppose now that $st > 0$.

Then $g(s+t) = g((1+\frac{t}{s})s) \le (1+\frac{t}{s})g(s)$ and $g(s+t) = g((1+\frac{s}{t})t) \le (1+\frac{s}{t})g(t)$.

Thus, by (i), $g(s+t) \le \min((1+\frac{t}{s})g(s), (1+\frac{t}{t})g(t)) \le g(s) + g(t)$.

Obviously the stronger condition $\lambda > 1 \Rightarrow g(\lambda s) < \lambda g(s)\ \forall\ s$ implies the strict inequality $g(s+t) < g(s) + g(t)$.

(ii). It is obvious that if g is a sub-additive function, then the function $h(s) = \alpha g(\lambda s) + \beta$ is again sub-additive for any $\lambda \in \Re$, $\alpha,\beta > 0$ (indeed, $h(s + t) = \alpha g(\lambda s + \lambda t) + \beta \le \alpha g(\lambda s) + \alpha g(\lambda t) + \beta \le h(s) + h(t)$). If we write $s = ax$, the symmetry condition (S) becomes $g(ax) + g(a(1-x)) = b \Leftrightarrow \frac{g(ax)}{b} + \frac{g(a(1-x))}{b} = 1$we see that the function g is sub-additive if and only if the function $\varphi(x) = \frac{g(ax)}{b}$ is sub-additive. It means that we can put in the condition (S) $a = b = 1$.

If $st \ge 0$ the sub-additivity has been already proved at (ii).

Suppose s *and* t have opposite signs, for example $s > 0$ and $t < 0$. We shall use the symmetry property (S).

Write $t = 1 - v$. Then $v > 1$. The claimed inequality becomes
$\varphi(s + 1 - v) < \varphi(s) + \varphi(1 - v) \Leftrightarrow 1 - \varphi(v-s) < \varphi(s) + 1 - \varphi(v) \Leftrightarrow \varphi(v) < \varphi(s) + \varphi(v-s)$.

This is true if $v - s \ge 0$ since $v = s + (v-s)$ and s and v-s are positive. So the trick holds if $s+t \le 1$. If $s + t > 1$, we write $s = 1 - u$, notice that $u - t < 0$ and write the claimed inequality as $\varphi(1-u+t) < \varphi(1-u) + \varphi(t) \Leftrightarrow 1 - \varphi(u-t) < 1 - \varphi(u) + \varphi(t) \Leftrightarrow \varphi(u-t) + \varphi(t) > \varphi(u)$ which is true again by Lemma 4. This ends the proof□

References

[1]P. Guijarro and M. S. Tomas : Perpendicular bisectors and orthogonality. Arch Math 69(1997) 491 – 496. Birkhauser, Basel 1997.

[2] Gyula Maksa and Peter Volkmann, Characterization of group homomorphisms having values in an inner product space, Publ. Math.Debrecen 56 (2000), 197-200.

[3] Rădulescu S. and Rădulescu M., On the Maksa-Volkmann conjecture, Talk at the 7th Congress of Romanian Mathematicians, Brasov, 2011.

[4] Cohen, Joel E.; Kemperman, J. H. B.; Zbăganu, Gh. Comparisons of stochastic matrices. With applications in information theory, statistics, economics, and population sciences. *Birkhäuser Boston, Inc., Boston, MA,* 1998.

[5] M.S.Tamas, Circumcenters in Real Normed Spaces, Bollettino dell'Unione Matematica Italiana, , Bollettino U.M.I. 8 -B 2005), pp 421-430, http://www.bdim.eu/item?fmt=pdf&id=BUMI_2005_8_8B_2_421_0

[6] Undine Leopold, Horst Martini, Geometry of simplices in Minkowski Spaces, November 17, 2018, https://arxiv.org/pdf/1710.04343.pdf

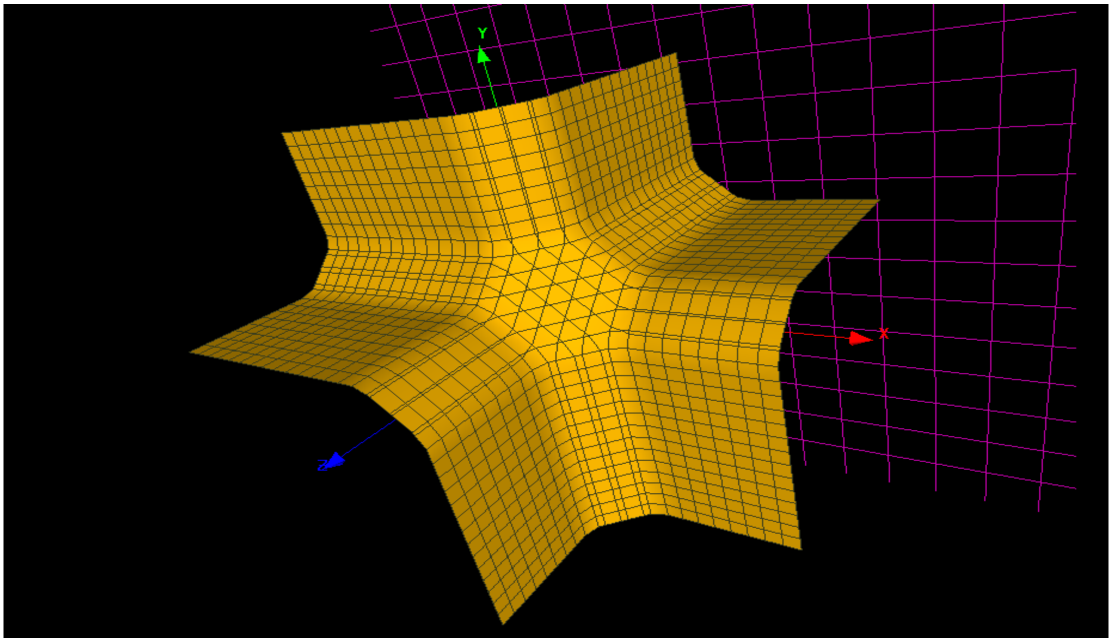
[7] Thomas Jahn, Margarira Spirova, On Bisectors in Normed Spaces, https://arxiv.org/abs/1409.1833

[8] A.G. Horvath, On Bisectors in Minkowski Normed Spaces, *Acta Mathematica Hungarica* volume 89, pages 233–246(2000)

[9] A. G. Horv́ath, Bisectors in Minkowski 3-space, Beitr̈age zur Algebra und Geometrie Contributions to Algebra and Geometry, Volume 45, No. 1, 225-238. (2004), http://emis.impa.br/EMIS/journals/BAG/vol.45/no.1/b45h1hor.pdf

[10] M.M.Day , Some characterizations of inner-product spaces, Trans Amer Mat Soc 62 (1947) 320-337 https://www.ams.org/journals/tran/1947-062-02/S0002-9947-1947-0022312-9/S0002-9947-1947-0022312-9.pdf

[11] Heinrich Mann: *Untersuchungen über Wabenzellen bei allgemeiner Minkowskischer Metrik*, Monatshefte fur Mathemathik Physik, 42, S. 417–424(1935)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