Time-domain moment matching for nonlinear dynamical systems of ODEs

Something to compute...

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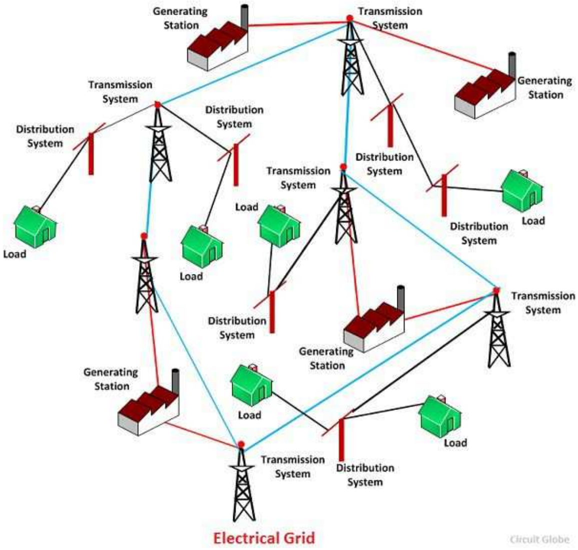
Example of complex large model: a power grid



Models–nonlinear interconnected ⇒ highly dimensional

Network modelling ⇒ large-scale models unsuitable for simulation and control.

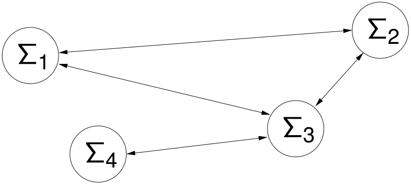
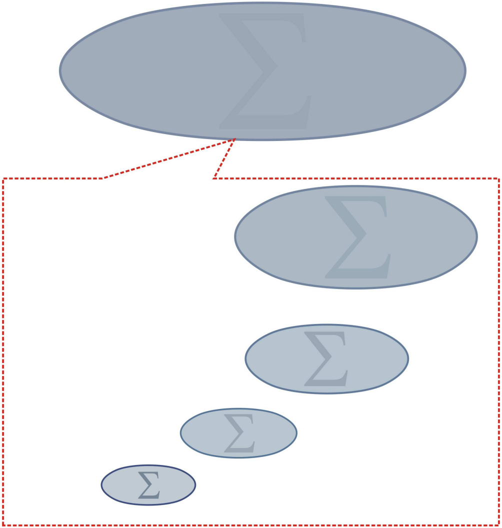


Figure: An example of a network system.

Arrows indicate the interactions between the subsystems $\Sigma_1, \Sigma_2, \Sigma_3$ and Σ_4.

Model reduction



Problem formulation

Given a dynamical system of ODEs

$$\begin{aligned} \dot{x} &= f(x, u, t), \quad x(0) = x_0, \\ y &= h(x, u, t), \end{aligned} \tag{1}$$

with the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$ and the output $y \in \mathbb{R}^p$,
find a system

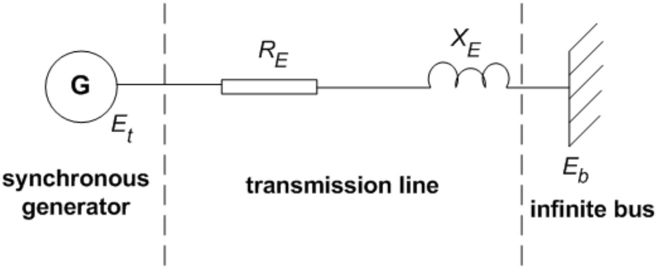
$$\begin{aligned} \dot{\xi} &= \varphi(\xi, u), \quad \xi(0) = \xi_0, \\ \chi &= \psi(\xi, u), \end{aligned} \tag{2}$$

with the state $\xi(t) \in \mathbb{R}^\nu$ and the output $\chi \in \mathbb{R}^p$, such that

- $\nu \ll n$,
- sys. (2) preserves the properties/structure of sys. (1),
- sys. (2) approximates sys. (1) within an error bound,
- system (2) is easy to compute.

Nonlinear model reduction - Motivation

Σ_is: mechanical systems, nonlinear electrical networks, power systems, waste incineration processes, etc.



Q: Why is model reduction necessary?

- Generator: 8 states = 6 currents + 1 angle + 1 velocity;
- Hard: analyze dissipativity/passivity; manipulate for control;
- *N* generators interconnected through transmission lines...

Nonlinear model reduction - Motivation

Q: What tools were available?

- Methods based on physical intuition (on ad-hoc basis): singular perturbation [Sauer & Ahmed & Zaid & Kokotovich '88), modal analysis [Meirovitch '70], POD, empirical balancing [Lall et all '02].
- Customary approach: model reduction of the linearized model; MOR of bilinear models [Benner et al. 20xx].
- Drawbacks:
  - *no* prior error bounds,
  - *lost* accuracy,
  - *lost* properties,
  - *lost* physical interpretation,
  - *lost* I/O interpretation (control!?).

Nonlinear model reduction - new methods

- Analytical methods based on balancing theory : extension of the linear tool [Scherpen ’94, Scherpen & Gray ’00, Fujimoto & Scherpen ’03, ’05, ’07, ’10] $\leftarrow$ no numerical tool yet :-(

- Methods based on a time-domain approach to the moment matching problem [Astolfi TAC2010, I & Astolfi CDC10, I & Astolfi ACC2013, I & Astolfi NOLCOS2013, I & Astolfi TAC2016] $\leftarrow$ something to compute, see later... (better ain’t it? :-))

  Moments: the solution of a (nonlinear) Sylvester equation; the steady-state of *signals*.

  The “interpolation” problem is *systemically* solved $\rightarrow$ *parametrized* reduced order models.

  Ingredients: signals & nonlinear systems, output regulation ideas $\leftarrow$ stuff from systems & control theory.

MM Motivation

Moment matching techniques for a complete overview see [Antoulas '05]:

- reduction of large scale systems (resulting from PDE discretizations, etc.);
- use projection methods $\Rightarrow$ efficient numerical tools;
- moment matching with preservation of properties $\leftarrow$ difficult choice of interpolation points.

Drawbacks for nonlinear:

- interpolation points as an s-domain notion, useless from a nonlinear p.o.v. $\rightarrow$ no direct nonlinear counterpart of the frequential interpolation;
- difficulty in choosing interpolation points for properties $\leftarrow$ the systemic property depends on the interpolation points;
- lack of a sound system theoretical interpretation (as opposed to balancing).

A new approach

Time-domain perspective $\rightarrow$ moment as the steady-state response of a system excited by a *chosen* signal generator.

MM model reduction problem: Given a *n*-th order system of ODEs find a ν order system of ODEs such that the steady-state responses at ν imposed inputs match!

Outcome:

- families of reduced order models that match moment of the system at given modes of the signal generator $\Leftrightarrow$ "interpolation" points;
- parameterized models $\rightarrow$ parameters enforce properties $\rightarrow$ stable models, passive approximations, port-Hamiltonian/gradient models, two-sided MM models...

The notion of moment [Astolfi TAC10]

Consider a **linear, SISO, minimal** system $\dot{x} = Ax + Bu$, $y = Cx$, with the associated transfer function (analytic using Laplace transform) $K(s) = C(sI - A)^{-1}B$. Take

$$s^* \in \mathbb{C} \setminus \sigma(A) \Rightarrow K(s) = K(s^*) - \frac{\mathrm{d}\,K}{\mathrm{d}\,s}(s^*)(s - s^*) + \frac{1}{2}\frac{\mathrm{d}^2\,K}{\mathrm{d}\,s^2}(s^*)(s - s^*)^2 + \ldots$$

The moments of the system at $s^* \in \mathbb{C}$ are $\eta_k(s^*) = \frac{(-1)^k}{k!}C(s^*I - A)^{-(k+1)}B$.

A closer look at the 0-moment at s^*: $\eta_0(s^*) = C\Pi$, where Π uniquely satisfies $A\Pi + B = \Pi s^*$.

Theorem (General result)

Consider ν interpolation points of the type s^*. Then the moments $\eta_k(s_{(\cdot)}), \ldots$ are in one-to-one relation with the matrix $C\Pi$, where Π is the unique solution of the Sylvester equation

$$A\Pi + BL = \Pi S$$

with S any real matrix with $\sigma(S) = \{\eta_k(s_{(\cdot)}), \ldots\}$, with ν elements and L any matrix such that the pair (L, S) is observable.

Moment matching [Astolfi TAC10]

Reduced order model:

$$\dot{\xi} = F\xi + Gu, \;\; \psi = H\xi + Du,$$

$\xi(t) \in \mathbb{R}^{\nu}$, $\widehat{K}(s) = \widehat{K}(s^*) - \frac{\mathrm{d}\,\widehat{K}}{\mathrm{d}\,s}(s^*)(s-s^*) + \frac{1}{2}\frac{\mathrm{d}^2\,\widehat{K}}{\mathrm{d}\,s^2}(s^*)(s-s^*)^2 + \ldots$ is a model of (A, B, C) at S, with $\sigma(S) \cap \sigma(A) = \emptyset$, if $\sigma(S) \cap \sigma(F) = \emptyset$ and

$$C\Pi = HP,$$

where Π is the unique solution of $A\Pi + BL = \Pi S$, with L chosen such that (L, S) is observable and P invertible is the unique solution of

$$FP + GL = PS.$$

Note that this model matches the moments of the original sys. at $\sigma(S)$.

For $P = I$, a family of reduced order models is

$$\dot{\xi} = (S - GL)\xi + Gu,$$
$$\psi = C\Pi\xi,$$

with G any matrix chosen such that $\sigma(S) \cap \sigma(S - GL) = \emptyset$ $\leftarrow$ models parameterized in G $\rightarrow$ used to enforce properties (see nonlinear case for details).

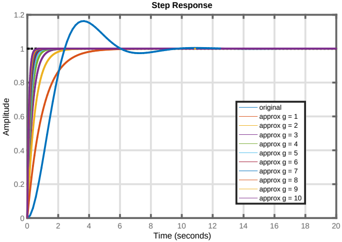
Illustrative example

$$\ddot{x} + \dot{x} + x = u, \; y = x \Leftrightarrow \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \; y =$$

$$x \Leftrightarrow K(s) = \frac{1}{s^2 + s + 1} \text{ and}$$

$$\dot{\xi} = -g\xi + gu, \; y = x \Leftrightarrow \widehat{K}(s) = \frac{g}{s+g}.$$

$$x = \xi = 1, \text{ for } t \to \infty \;\&\; u(t) = 1, t \geq 0, \forall g.$$



$K(0) = \widehat{K}(0) = 1, \; \forall g \Rightarrow \widehat{K}$ matches the moment of K at 0, $\forall g$.

In the nonlinear case, $K(s)$ does not exist, all done in the time-domain.

Example: Pendulum: $\ddot{\theta} + \dot{\theta} + \sin\theta = u$.

Notion of moment [Astolfi CDC2008, Astolfi TAC2010]

SISO nonlinear system

$$\dot{x} = f(x) + g(x)u, \;\; y = h(x), \quad x(t) \in \mathbb{R}^n,$$

with $f(\cdot), g(\cdot), h(\cdot), d(\cdot)$ smooth mappings.

Signal generator

$$\dot{\omega} = s(\omega), \;\; \theta = l(\omega),$$

with $\omega(t) \in \mathbb{R}^\nu$ and $s(\cdot)$ and $l(\cdot)$ smooth. Suppose that $f(0) = 0$, $h(0) = 0$, $s(0) = 0$ and $l(0) = 0$ & 2 working assumptions.

Assumption

$\exists!$ mapping $\pi(\omega)$ solving the PDE:

$$\frac{\partial \pi(\omega)}{\partial \omega} s(\omega) = f(\pi(\omega)) + g(\pi(\omega)) l(\omega). \tag{3}$$

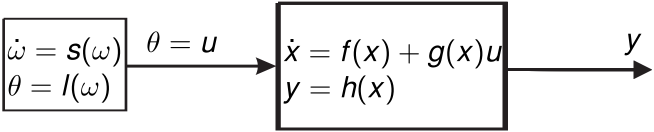
$\leftarrow$ interconnection $u = \theta$ has $x = \pi(\omega)$ invariant manifold; system restricted to the invariant manifold is described by $\dot{\omega} = s(\omega)$.

Notion of moment [Astolfi CDC2008, Astolfi TAC2010]

Assumption

Signal generator yields persistent signals.

$\leftarrow$ from the theory of nonlinear output regulation and nonlinear steady-state response.



Definition (Nonlinear moment)

$h(\pi(\omega))$: moment of the nonlinear system @ $\{s(\omega), l(\omega)\}$.

Nonlinear moment matching [Astolfi TAC2010]

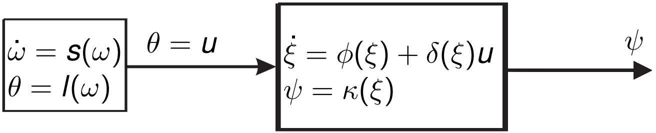
Theorem (Nonlinear moment & steady-state response)

Assume 0 *is a locally exp. stable eq. The moment of the nonlinear system is the steady-state response of the interconnection of the system with the signal generator.*

Reduced order model:

$$\dot{\xi} = \phi(\xi) + \delta(\xi)u, \;\; \psi = \kappa(\xi),$$

$\xi(t) \in \mathbb{R}^{\nu}$: a model @ $\{s(\omega), l(\omega)\}$ if the moment is the same.



Lemma (Sufficient condition for matching)

$\exists p(\cdot)$ *uniquely satisfying* $\frac{\partial p}{\partial \omega} s(\omega) = \phi(p(\omega)) + \delta(p(\omega)) l(\omega)$ *s.t.* $h(\pi(\omega)) = \kappa(p(\omega))$.

Family of parameterized models [Astolfi TAC2010]

For $p(\omega) = \omega (= \xi) \Rightarrow \kappa(\xi) = h(\pi(\xi)) \leftarrow p(\omega) = \xi$ coordinate transformation.

Family of **reduced order** models that achieve moment matching @ $\{s(\omega), l(\omega)\}$:

$$\dot{\xi} = s(\xi) - \delta(\xi) l(\xi) + \delta(\xi) u,$$
$$\psi = h(\pi(\xi)),$$

where $\delta(\cdot)$ is such that the equation

$$\phi(p(\omega)) + \delta(p(\omega)) l(\omega) = \frac{\partial p(\omega)}{\partial \omega} s(\omega),$$

with $\phi(\xi) = s(\xi) - \delta(\xi) l(\xi)$, has the unique solution $p(\omega) = \omega$.

Application: PH system

Port Hamiltonian systems:

- Physical systems $\leftarrow$ mechanical systems, electrical circuits, electromechanical systems, power systems, chemical processes,...
- $\dot{x} = (J(x) - R(x))\dfrac{\partial \mathcal{H}(x)}{\partial x} + g(x)u, \quad y = g^T(x)\dfrac{\partial \mathcal{H}(x)}{\partial x},$

- $x(t) \in \mathbb{R}^n$;

- $J(x) = -J^*(x)$ - interconnection matrix;

- $R(x) = R^*(x) \geq 0$ - dissipation matrix;

- the Hamiltonian $\mathcal{H}(x) : \mathbb{R}^n \to \mathbb{R}$ $\leftarrow$ energy, co-energy,...

- $g(x) \in \mathbb{R}^n$;

- all mappings are assumed smooth.

Example - double-mass, double-spring and damper system of order 4

Compute the first order port Hamiltonian model matching the moment of the system at $\{0, \omega\}$.

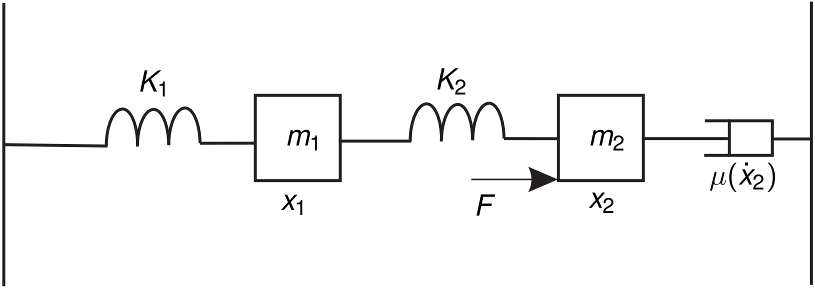


Figure: Double-mass, double-spring and damper system

Example - double-mass, double-spring and damper system of order 4 contd.

$m_1 = m_2 = 1$, $k_1 = k_2 = 1$, $F = \dot{x}_2 u$, $\mu(\dot{x}_1) = x_2^2 + 1 \Rightarrow$ nonlinear port Hamiltonian model:

$$J(x) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad R(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu(x_4) \end{bmatrix},$$

$$g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ x_4 \end{bmatrix}, \quad H(x) = \frac{1}{2}(x_3^2 + x_4^2) + \frac{1}{2}(x_2 - x_1)^2 + \frac{1}{2}x_1^2,$$

$$y = x_2 + x_4^2.$$

PH reduced order model [I. & Astolfi NOLCOS2013, I & Astolfi TAC202x–ongoing]

Signal generator $\{s(\omega), l(\omega)\},\ \omega \in \mathbb{R}^\nu$; mapping π.

Port Hamiltonian reduced model of order ν:

$$\dot{\omega} = (\bar{J}(\omega) - \bar{R}(\omega))\frac{\partial^T \bar{\mathcal{H}}(\omega)}{\partial \omega} + \bar{g}(\omega)u,\ \psi = \bar{g}^T(\omega)\frac{\partial^T \bar{\mathcal{H}}(\omega)}{\partial \omega}, \quad (4)$$

where

$$\bar{J}(\omega) = \left.\frac{\partial \rho(x)}{\partial x}\right|_{x=\pi(\omega)} J(\pi(\omega)) \left.\frac{\partial \rho^T(x)}{\partial x}\right|_{x=\pi(\omega)},$$

$$\bar{R}(\omega) = \left.\frac{\partial \rho(x)}{\partial x}\right|_{x=\pi(\omega)} R(\pi(\omega)) \left.\frac{\partial \rho^T(x)}{\partial x}\right|_{x=\pi(\omega)},$$

$$\bar{g}(\omega) = \left.\frac{\partial \rho(x)}{\partial x}\right|_{x=\pi(\omega)} g(\pi(\omega)),\ \bar{\mathcal{H}}(\omega) = \mathcal{H}(\pi(\omega)),$$

with $\rho : \mathbb{R}^n \to \mathbb{R}^\nu$.

PH reduced order model [I. & Astolfi NOLCOS2013, I. & Astolfi TAC202x–ongoing]

Theorem

If

1. $\rho(\pi(\omega)) = \omega$,

2. $\dfrac{\partial \bar{\mathcal{H}}(\omega)}{\partial \omega} \dfrac{\partial \rho(x)}{\partial x}\Big|_{x=\pi(\omega)} g(\pi(\omega)) = \dfrac{\partial \mathcal{H}(x)}{\partial x}\Big|_{x=\pi(\omega)} g(\pi(\omega))$,

then (4) (parameterized in ρ) matches the moment $g^T(\pi(\omega))\dfrac{\partial^T \mathcal{H}(x)}{\partial x}\Big|_{x=\pi(\omega)}$ of the pH system at $\{s(\omega), l(\omega)\}$.

PH ROMs $\in$ the family of ROMs achieving moment matching [I. & Astolfi TAC202x–ongoing]

The models $\Sigma_{\pi(\omega)}$ are a subset of the family of models $\Sigma_{\delta(\xi)}$ described by

$$\dot{\xi} = s(\xi) - \delta(\xi) l(\xi) + \delta(\xi) u, \ \eta = \psi(\xi),$$

for particular instances of $\delta(\xi)$.

Corollary

Consider the class of ν order systems $\Sigma_{\delta(\xi)}$.
$\Sigma_{\delta(\omega)}$ is a reduced order port Hamiltonian system iff we pick

$$\delta(\omega) = \left. \frac{\partial \rho(x)}{\partial x} \right|_{x=\pi(\omega)} g(\pi(\omega)).$$

Example continued

Let $s(\omega) = 0,\ l(\omega) = \omega$, and let $\pi(\omega) = [\pi_1(\omega)\ \pi_2(\omega)\ \pi_3(\omega)\ \pi_4(\omega)]^T,\ \pi(\omega) : \mathbb{R} \to \mathbb{R}^4$, with

$$\pi_1(\omega) = \pi_2(\omega)/2 = \omega^3 - \omega^2 + \omega, \pi_3(\omega) = 0, \pi_4(\omega) = -\omega.$$

A first order port Hamiltonian system matching the moment $\omega^3 + \omega$ @ $\{0, \omega\}$; $\bar{\mathcal{H}}(\omega) = \frac{1}{2}\omega^2 + (\omega^3 - \omega^2 + \omega)^2$.

1st order port Hamiltonian model matching $\omega^3 + \omega$ @ $\{0, \omega\}$: $\bar{J}(\xi)0$, $\bar{R}(\xi) = (\omega^2+1)^3/(\omega^2\beta^2(\omega))$ & $\bar{g}(\xi) = (\omega^2+1)/\beta(\omega)$, with $\beta(\omega) = 6\omega^4 - 10\omega^3 + 12\omega^2 - 6\omega + 3 = \bar{\mathcal{H}}'(\omega)$, i.e.,

$$\dot{\omega} = \frac{(\omega^2+1)^3}{\omega\beta(\omega)}(u-1), \quad \psi = \omega^3 + \omega,$$

belonging to the family of first order models $\dot{\xi} = -\delta(\xi)\xi + \delta(\xi)u, \psi = h(\pi(\xi))$, with $\delta(\xi) = (\xi^2+1)^3/(\xi\beta(\xi))$.

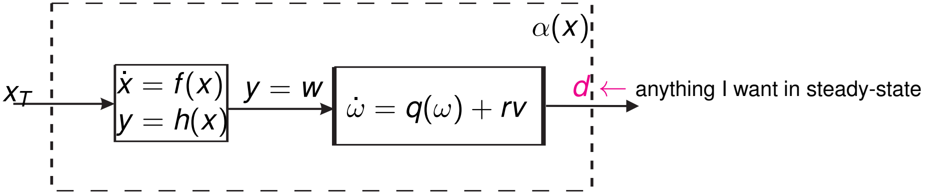
Signal generator driven by the system–”swapped” interconnection

Generalized signal generator:

$$\dot{\omega} = q(\omega) + rv, \ \omega(0) = 0, \ d = \omega + \boldsymbol{v}(x),$$

with $\omega(t) \in \mathbb{R}^\nu$, $q : \mathbb{R}^\nu \to \mathbb{R}^\nu$ a smooth mapping, $q(0) = 0$, $r \in \mathbb{R}^\nu$, $\boldsymbol{v} : \mathbb{R}^n \to \mathbb{R}^\nu$ a smooth mapping, $\boldsymbol{v}(0) = 0$, $w(t) \in \mathbb{R}$ and $d(t) \in \mathbb{R}^\nu$.

Consider the interconnection depicted in the figure.



Assume that the system (f, g, h) is observable and the signal generator is reachable from $\omega(0) = 0$.

Define $\alpha : \mathbb{R}^n \to \mathbb{R}^\nu$, $\omega_T = \alpha(x_T)$, for all T, $\alpha(0) = 0$. Hence $d(T) = \alpha(x_T) + \boldsymbol{v}(x_T)$.

Behaviour of the signal $d(t)$ [I. & Astolfi TAC2016]

Lemma

Let $d(t)$ be differentiable. Then $d(t)$ obeys the dynamics

$$\dot{d} = q(\omega) - q(\omega - d) + \frac{\partial \boldsymbol{\upsilon}(x)}{\partial x} g(x)\bigg|_{x=\rho(d-\omega)} u,$$

where $\rho : \mathbb{R}^{\nu} \to \mathbb{R}^{n}$ is a mapping such that $\rho(\boldsymbol{\upsilon}(x)) = x$, iff $\boldsymbol{\upsilon}(\cdot)$ satisfies the equation

$$-q(-\boldsymbol{\upsilon}(x)) = \frac{\partial \boldsymbol{\upsilon}(x)}{\partial x} f(x) + rh(x).$$

$\leftarrow$ new Sylvester equation which helps define a new notion of moments and moment matching.

Moment–steady state of $d(t)$ [I. & Astolfi TAC2016]

Working assumption The mapping $\boldsymbol{\upsilon}$ is the unique solution of the nonlinear partial differential Sylvester equation.
$\exists \rho : \mathbb{R}^\nu \to \mathbb{R}^n$ such that $\rho(\boldsymbol{\upsilon}(x)) = x$.

Definition

We call the moment of at $\{q(\omega), r\}$: $\left.\frac{\partial \boldsymbol{\upsilon}(x)}{\partial x} g(x)\right|_{x=\rho(d-\omega)}$.

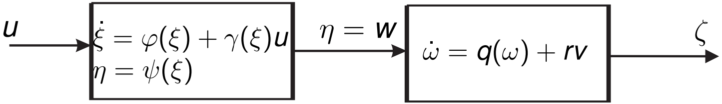
Theorem

Assume the origin is a locally exponentially stable equilibrium. Then the moment @ $\{q(\omega), r\}$ is in one-to-one relation with the steady-state of $d(t)$ for $u = \delta_0(t)$ (Dirac function).

Moment matching–new [I. & Astolfi TAC2016]

Nonlinear system (φ, γ, ψ): $\dot{\xi} = \varphi(\xi) + \gamma(\xi)u$, $\eta = \psi(\xi)$, where $\xi(t) \in \mathbb{R}^\nu$, $u(t) \in \mathbb{R}$ and $\eta(t) \in \mathbb{R}$, with φ, γ and ψ smooth.

Consider the interconnection $w(t) = \eta(t)$, with the output $\zeta(t) \in \mathbb{R}^\nu$.



Definition

System (φ, γ, ψ) matches the moment @ $\{q(\omega), r\}$ if the steady-state response of the output signal $\zeta(t)$ matches the steady-state response of the signal $d(t)$.

Theorem (Moment matching)

Let $\zeta = \omega + \xi$. Then (φ, γ, ψ) matches the moment @ $\{q(\omega), r\}$ iff $\varphi(\xi) = -q(-\xi) - r\psi(\xi)$ and $\gamma(\xi) = \left.\frac{\partial v(x)}{\partial x} g(x)\right|_{x=\rho(\xi)}$, for all $\psi(\xi) \in \mathbb{R}$.

Equations

$$\Sigma_{\psi(\xi)} : \begin{cases} \dot{\xi} = -q(-\xi) - r\psi(\xi) + \left.\frac{\partial v(x)}{\partial x} g(x)\right|_{x=\rho(\xi)} u, \\ \eta = \psi(\xi), \end{cases}$$

with $\xi(t) \in \mathbb{R}^\nu$ define a family of models of order ν, parameterized in $\psi(\xi) \in \mathbb{R}$, matching the moment @ $\{q(\omega), r\}$.

Linear case: System (A, B, C) of order n. Moments @ interpolation points s_i, $i = 1, \ldots, \nu$: $C(s_i I - A)^{-1} B = \Upsilon B$, where Υ uniquely satisfies the dual Sylvester equation $Q\Upsilon = \Upsilon A + RC$, $s_i \in \sigma(Q)$, $Q \in \mathbb{C}^{\nu \times \nu}$, (Q, R) controllable. Moments related to the steady-state of $d(t)$: $\dot{d} = Qd + \Upsilon Bu$. System (F, G, H): $\dot{\xi} = F\xi + Gu$, $\eta = H\xi$, $\xi(t) \in \mathbb{R}^\nu$ matches the moments of (A, B, C) at $\sigma(Q)$ iff the the output ζ satisfies $\dot{\zeta} = Q\zeta + \Upsilon Bu$, iff $F = Q - RH$, $G = \Upsilon B$.

Relations between the families of models [I. & Astolfi TAC2016]

Theorem (Equivalence of the families of models)

The following statements are equivalent.

1. $\Sigma_{\psi(\xi)}$ *matches the moments of* (f, g, h) *at* $\{q(\omega), r\}$.
2. $r^T \psi(\xi) = \left.\dfrac{\partial \boldsymbol{\upsilon}(x)}{\partial x}\right|_{x=\rho(\xi)}, \; \psi(\omega) = h(\pi(\omega))$.
3. $\Sigma_{\psi(\xi)}$ *matches the moments of* (f, g, h) *at* $\{-q(-\omega), r^T\}$.
4. $\Sigma_{\delta(\xi)}$ *matches the moments of* (f, g, h) *at* $\{-q(-\omega), r^T\}$.
5. $r^T h(\pi(\xi)) = \delta^T(\xi), \; \delta^T(\xi) = \left.\dfrac{\partial \boldsymbol{\upsilon}(x)}{\partial x}\right|_{x=\rho(\xi)}$.
6. $\Sigma_{\delta(\xi)}$ *matches the moments of* (f, g, h) *at* $\{q(\omega), r\}$.

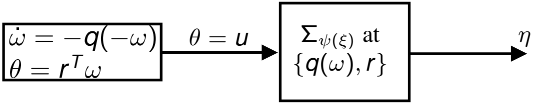


Figure: Diagram illustrating statements 1), 2) and 3) of the Theorem.

Matching two-moments–illustrative example [I. TAC2016]

Let $\eta_0 \in \mathbb{C}$ and $\eta_1 \in \mathbb{C}$ be such that $\eta_0 \neq \eta_1$. The transfer function

$$K_g(s) = \frac{\eta_0 g}{s - s_0 + g},$$

with $g \in \mathbb{C}$, defines a class of 1st order models, parameterized in g, that match the moment η_0 at $s_0 \in \mathbb{C}$, i.e., $K_g(s_0) = \eta_0$.

Let $q_1 \in \mathbb{C}$ be such that $q_1 \neq s_0$.

Aim: 1st order model K_g that also matches the moment η_1 at q_1, i.e., compute the parameter g, such that
$K_g(q_1) = \eta_1 \Leftrightarrow \eta_0 g = [(s_0 - q_1) + g]\eta_1 \Leftrightarrow g(\eta_0 - \eta_1) = (s_0 - q_1)\eta_1$.
Since we assume $\eta_0 \neq \eta_1$, we obtain the unique parameter

$$g = \eta_1 \frac{s_0 - q_1}{\eta_0 - \eta_1}.$$

Let $p = \dfrac{\eta_0 - \eta_1}{s_0 - q_1}$. Note that since we assume that $s_0 \neq q_1$, p is well defined. Hence $g = \eta/p$. Since g is unique, then p is also unique and it satisfies the equation $ps_0 - q_1 p = \eta_0 - \eta_1$.

Proposition

1. $\exists$ *a subfamily of models* $\Sigma_{\delta(\xi)}$ *matching the moment @* $\{s(\omega), l(\omega)\}$ *and @* $\{q(\omega), r\}$ *simultaneously, iff* $\delta(\xi)$ *satisfies the equation* $\frac{\partial p(\xi)}{\partial \xi}\delta(\xi) = \frac{\partial \boldsymbol{v}(x)}{\partial x} g(x)\Big|_{x=\rho(\xi)}$, *where* $p : \mathbb{R}^\nu \to \mathbb{R}^\nu$ *satisfies the equation*
$$q(-p(\xi)) + \frac{\partial p(\xi)}{\partial \xi} s(\xi) = \frac{\partial \boldsymbol{v}(x)}{\partial x} g(x)\Big|_{x=\rho(\xi)} l(\xi) - rh(\pi(\xi)).$$
2. $\exists$ *a subfamily of models* $\Sigma_{\psi(\xi)}$ *matching the moment @* $\{s(\omega), l(\omega)\}$ *and @* $\{q(\omega), r\}$ *simultaneously, iff* $\psi(p(\omega)) = h(\pi(\omega))$, *where* $p : \mathbb{R}^\nu \to \mathbb{R}^\nu$ *satisfies the equation*
$$q(-p(\omega)) + \frac{\partial p(\omega)}{\partial \omega} s(\omega) = \frac{\partial \boldsymbol{v}(x)}{\partial x} g(x)\Big|_{x=\rho(p(\omega))} l(\omega) - rh(\pi(\omega)).$$

If p is identity, then $\Sigma_{\delta(\xi)} = \Sigma_{\frac{\partial \boldsymbol{v}(x)}{\partial x} g(x)\big|_{x=\rho(\xi)}} = \Sigma_{h(\pi(\xi))} = \Sigma_{\psi(\xi)}$ is the unique model matching the moment @ $\{s(\omega), l(\omega)\}$ and @ $\{q(\omega), r\}$, simultaneously.

A practical example: DC-to-DC Ćuk converter

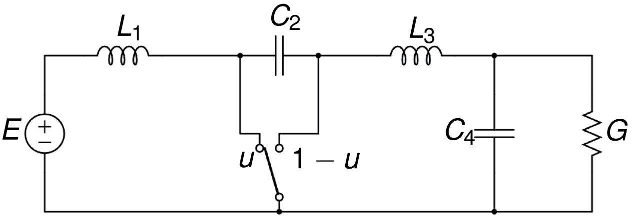


Figure: DC-to-DC Ćuk converter circuit.

Averaged model [Rodriguez-Ortega-Astolfi-ACC2005]:

$$L_1 \dot{i}_1(t) = -(1-u)v_2 + E, \quad C_2 \dot{v}_2(t) = (1-u)i_1 + ui_3,$$
$$L_3 \dot{i}_3(t) = -uv_2 - v_4, \quad C_4 \dot{v}_4(t) = i_3 - Gv_4, \quad y = v_4,$$

where $i_1(t) \in \mathbb{R}^+$ and $i_3(t) \in \mathbb{R}^-$ currents, $v_2(t) \in \mathbb{R}^+$ and $v_4(t) \in \mathbb{R}^-$ voltages, L_1, C_2, L_3, C_4 and G positive parameters, $E \in \mathbb{R}$ and $u(t) \in (0,1)$ continuous control signal = the slew rate of a PWM circuit used to control the switch position in the converter.

Approximating DC-to-DC Ćuk converter (1)

Signal generator: $q(\varpi, v) = \varpi + v$, output $d = \varpi + \boldsymbol{\upsilon}(x)$.

State $x^T = [x_1 \ x_2 \ x_3 \ x_4]^T = [i_1 \ v_2 \ i_3 \ v_4]^T \in \mathbb{R}^4$.

Sylvester eq.:

$$\frac{\partial \boldsymbol{\upsilon}(x)}{\partial x} f(x, 0) + C_4 x_4 = 0,$$

with solution

$$\boldsymbol{\upsilon}(x) = \frac{L_3}{C_4 L_3 + G L_3 - 1} x_3 + \frac{C_4 L_3}{C_4 L_3 + G L_3 - 1} x_4 + \text{h.o.t.}$$

Family of first order models matching the moment at $C_4 v$:

$$\begin{aligned} \dot{\xi} &= -C_4 \psi(\xi) - \frac{v_2}{C_4 L_3 + G L_3 - 1} u, \\ \eta &= \psi(\xi), \end{aligned} \qquad (5)$$

parametrized in ψ and v_2.

Approximating DC-to-DC Ćuk converter (2)

Signal generator $\{s(\omega), l(\omega)\} = \{0, \omega\}$.

Sylvester eq unique solution

$$\pi(\omega) = \begin{bmatrix} GE\frac{\omega^2}{(1-\omega)^2} \\ E\frac{1}{1-\omega} \\ GE\frac{\omega}{\omega-1} \\ E\frac{\omega}{\omega-1} \end{bmatrix}.$$

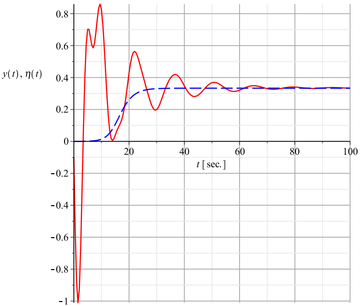
Moment: $h(\pi(p(\omega))) = \dfrac{p(\omega)}{p(\omega)-1}$.

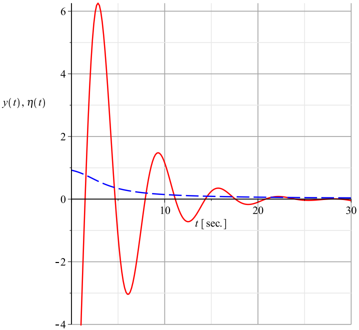
Unique first order model matching the moment at $\varpi + v$ and the moment at $\{0, \omega\}$, simultaneously:

$$\begin{aligned} \dot{\omega} &= -\omega - \frac{\omega}{\omega - 1}(u+1), \\ \eta &= \frac{\omega}{\omega - 1}. \end{aligned} \tag{6}$$

DC-to-DC Ćuk conv. approximation–simulations

Input $u(t) = \frac{1(t)}{2}$ v. input $u(t) = \epsilon e^t$, with $\epsilon > 0$.





Conclusions

MM for nonlinear systems:

- notion of moment for nonlinear systems as steady-state of the output when excited by a chosen signal generator;
- family of parametrized models matching the moment;
- free parameters determine subfamilies of stable, passive, more accurate approximations;
- free parameters determine subfamilies of port-Hamiltonian, gradient approximations;
- notion of moment as steady-state of a signal built from a signal generator driven by the system;
- a "dual" family of parametrized models matching the moment;
- two-sided moment matching: a subfamily of approximations. Under additional assumption $\Rightarrow$ unique model matching moments simultaneously.

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